Higher categories of n-bordisms

by

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Contents

Declaration

I hereby declare that the thesis entitled 'Higher categories of n-bordisms' submitted by me for the award of MS degree of the Indian Institute of Science did not form the subject matter for any other thesis submitted by me for any degree or diploma.

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Abstract

Higher category theory is the generalization of category theory to a context where there are not only morphisms between objects, but generally k-morphisms between $(k-1)$-morphisms, for all $k \in \mathbb{N}$. The theory of higher categories or $(\infty, 1)$-categories, as it is sometimes called, however, can be very intractable at times. That is why there are now several models which allow us to understand what a higher category should be. Among these models is the theory of quasi-categories, introduced by Bordman and Vogt, and much studied by Joyal and Lurie. There are also other very prominent models such as simplicial categories (Dwyer and Kan), relative categories (Dwyer and Kan), and Segal categories (Hirschowitz and Simpson). One of those models, complete Segal spaces, was introduced by Charles Rezk in his seminal paper "A model for the homotopy theory of homotopy theory". Later they were shown to be a model for $(\infty, 1)$-categories.

Higher bordism categories. One major application of higher category theory and one of the driving forces in developing it has been extended topological quantum field theory. This has recently led to what may become one of the central theorems of higher category theory, the proof of the cobordism hypothesis, conjectured by Baez and Dolan. Lurie suggested passing to (∞, n)-categories for a proof of the Cobordism Hypothesis in arbitrary dimension n. However, finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these n-dimensional TFTs, which are called "fully extended". Our focus will be on the (∞, d)-category $\mathrm{Bord}_n^{(n-d)}$, a variant of the fully extended Bord_n. Our goal is to sketch a detailed construction of the (∞, d)-category of n-bordisms as a d-fold complete Segal space, motivated by the proof due to Damien Calaque and Claudia Scheimbauer [CS19].

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I was extremely lucky to be a part of the category theory community at IISc, which includes students, faculty, postdoctoral fellows and visiting scholars. I learnt a great deal from attending the CoCT Seminars, and from countless discussions with my fellow senior, Subhajit Das.

Casting an even wider net, I have learnt much from the Indian category theory community, and have benefited greatly from discussions and collaborations with Rekha Santhanam from IIT Bombay and Srikanth Pai from ICTS Bengaluru (now at IISc).

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Introduction

Outline of the thesis

Topological field theories (TFTs) arose as toy models for physical quantum field theories and have proven to be of mathematical interest, notably because they are a fruitful tool for studying topology. An n-dimensional TFT is a symmetric monoidal functor from the category of bordisms, which has closed $(n-1)$-dimensional manifolds as objects and n-dimensional bordisms as morphisms, to any other symmetric monoidal category, which classically is taken to be the category of vector spaces or chain complexes.

A classification of 1- and 2-dimensional TFTs follows from classification theorems for 1- and 2-dimensional compact manifolds with boundary, cf. [Abr96]. In order to obtain a classification result for larger values of n one needs a suitable replacement of the classification of compact n-manifolds with boundary used in the low-dimensional cases. Moreover, as explained in [BD95], this approach requires passing to "extended" topological field theories. Here extended means that we need to be able to evaluate the n-TFT not only at n- and $(n-1)$-dimensional manifolds, but also at $(n-2)$-,..., 1-, and 0-dimensional manifolds. Thus, an extended n-TFT is a symmetric monoidal functor out of a higher category of bordisms. In light of the hope of computability of the invariants determined by an n-TFT, e.g. by a triangulation, it is natural to include this data. Furthermore, Baez and Dolan conjectured that, analogously to the 1-dimensional case, extended n-TFTs are fully determined by their value at a point, calling this the Cobordism Hypothesis. A definition of a suitable bicategory of n-bordisms and a proof of a classification theorem of extended TFTs for dimension 2 was given in [SP09].

In his expository manuscript [Lur09c], Lurie suggested passing to (∞,n)-categories for a proof of the Cobordism Hypothesis in arbitrary dimension n. He gave a detailed sketch of such a proof using a suitable higher category of bordisms, which, informally speaking, has zero-dimensional manifolds as objects, bordisms between objects as 1-morphisms, bordisms between bordisms as 2-morphisms, etc., and for $k > n$ there are only invertible k-morphisms given by diffeomorphisms and their isotopies. However, finding an explicit model for such a higher category poses one of the difficulties in rigorously defining these n-dimensional TFTs, which are called "fully extended".

In [Lur09c], Lurie gave a short sketch of a definition of this (∞,n)-category using complete n-fold Segal spaces as a model. Instead of using manifolds with corners and gluing them, his approach was to conversely use embedded closed (not necessarily compact) manifolds, following along the lines of [GTMW09, Gal11, BM14], and to specify points where they are cut into bordisms of which the embedded manifold is a composition. Whitney's embedding theorem ensures that every n-dimensional manifold M can be embedded into some large enough vector space and suitable versions for manifolds with boundary can be adapted to obtain an embedding theorem for bordisms. Moreover, the rough idea behind the definition of the n-fold Segal space is that it includes the data, for $k_1,\dots,k_n$, of the levels of PBord_n is that the $(k_1,\dots,k_n)$-level of our n-fold Segal space PBord_n should be a classifying space for diffeomorphisms of, in the ith direction k_i-fold, composable

n-bordisms. Lurie's idea was to use the fact that the space of embeddings of M into $\mathbb{R}^\infty$ is contractible to justify the construction.

Modifying this approach, the main goal of this report is to provide a detailed construction of (∞, d)-category of n-bordisms, a variant of (∞, n)-category of n-bordisms Bord_n suitable for explicitly constructing an example of a fully extended nTFT, which will be the content of a subsequent paper [Sch14].

Organization of the thesis

In Chapter 1, consisting of the first three sections, we recall the necessary tools from higher category theory needed to construct higher categories of bordisms.

Section 1.1 reviews the theory of simplicial sets and simplicial spaces, forming the basic objects of higher category theory. Section 1.2 reviews a model for $(\infty, 1)$-categories given by complete Segal spaces. In Section 1.3 we explain the model for (∞, n)-categories given by complete n-fold Segal spaces and introduce a model which is a hybrid between complete n-fold Segal spaces and Segal n-categories.

Chapter 2 is devoted to the construction of Bord_n.

Our construction of the (∞, d)-category $\mathrm{Bord}_n^{(n-d)}$ of higher bordisms is based on a simpler complete Segal space Int of closed intervals, which we introduce in Section 2.1. The closed intervals correspond to places where we are allowed to cut the manifold into the bordisms it composes. The fact that we prescribe closed intervals instead of just a point corresponds to fixing collars of the bordisms.

Section 2.2 is the central part of this thesis and consists of the construction of the complete d-fold Segal space $\mathrm{Bord}_n^{(n-d)}$ of n-bordisms, a variant of the n-fold Segal space Bord_n of n-bordisms.

Chapter 1

Complete n-fold Segal spaces

Section 1.1
A quick tour of simplicial spaces

In this section we take a quick look at categories and topological spaces to see how both of them can be thought of as special cases of simplicial sets. This is an informal review of these subjects, especially motivated by the Rasekh's exposition [Ras18] and serves as a motivation for our definition of a higher category, rather than a thorough introductory text. The section culminates in a introduction to simplicial spaces, which combines category theory and homotopy theory.

1.1.1 Review of Category Theory

The philosophy of categories is not to just focus on objects but also consider how they are related to each other. This leads to following definition of a category.

Definition 1.1.1. A category $\mathscr{C}$ is a set of objects $\mathscr{O}$ and a set of morphisms $\mathscr{M}$ along with following functions:

(1) An identity map $id : \mathscr{O} \to \mathscr{M}$.

(2) A source-target map $(s,t) : \mathscr{M} \to \mathscr{O} \times \mathscr{O}$.

(3) A composition map $m : \mathscr{M}^s \times^t_{\mathscr{O}} \mathscr{M} \to \mathscr{M}$.

These functions have to make the following diagrams commute:

(1) *Source-Target Preservation:*

$$\begin{tikzcd}
\mathscr{M} \arrow[d, "s"] & \mathscr{M} \underset{\mathscr{O}}{\times} \mathscr{M} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "m"] & \mathscr{M} \arrow[d, "t"] \\
\mathscr{O} & \mathscr{M} \arrow[l, "s"] \arrow[r, "t"] & \mathscr{O}
\end{tikzcd}$$

(2) *Identity Relations:*

$$\begin{tikzcd}
\mathscr{O} \arrow[r, "id"] \arrow[dr, "(id_{\mathscr{O}}, id_{\mathscr{O}})"'] & \mathscr{M} \arrow[d, "(s,t)"] \\
& \mathscr{O} \times \mathscr{O}
\end{tikzcd}$$

(3) *Identity Composition:*

$$\begin{tikzcd}
\mathscr{O} \times \mathscr{M} \arrow[r, "id \times id_{\mathscr{M}}"] \arrow[dr, "\pi_2"'] & \mathscr{M} \times \mathscr{M} \arrow[d, "m"] & \mathscr{M} \times \mathscr{O} \arrow[l, "id_{\mathscr{M}} \times id"'] \arrow[dl, "\pi_1"] \\
& \mathscr{M} &
\end{tikzcd}$$

(4) *Associativity:*

$$\begin{tikzcd}
\mathscr{M} \underset{\mathscr{O}}{\times} \mathscr{M} \underset{\mathscr{O}}{\times} \mathscr{M} \arrow[r, "id_{\mathscr{M}} \times m"] \arrow[d, "m \times id_{\mathscr{M}}"] & \mathscr{M} \underset{\mathscr{O}}{\times} \mathscr{M} \arrow[d, "m"] \\
\mathscr{M} \underset{\mathscr{O}}{\times} \mathscr{M} \arrow[r, "m"] & \mathscr{M}
\end{tikzcd}$$

There are many examples of categories in the world of mathematics.

Example 1.1.2. Let $\mathfrak{Set}$ be the category which has as objects all sets and as morphisms all functions of sets. Then the function *id* assigns to each set the identity function and the source target maps (s,t) assigns to each function it's source and target. Finally m is just the usual composition of functions.

Example 1.1.3. We can repeat the same example as above by replacing sets with a set that has additional structure. So, we can define the category $\mathcal{T}op$ of topological spaces and continuous maps, or groups and homomorphisms.

Remark 1.1.4. Very often we care about the morphisms between two specific objects. Concretely, for two objects $c, d \in \mathscr{C} = (\mathscr{O}, \mathscr{M})$ we want to define the set of maps with source c and target d and denote it as $Hom_{\mathscr{C}}(c,d)$, which we define as the following pullback

$$Hom_{\mathscr{C}}(c,d) = *^c \underset{\mathscr{O}}{\times}{}^s \mathscr{M}^t \underset{\mathscr{O}}{\times}{}^d *$$

Using the philosophy of categories on categories themselves means we should consider studying maps between categories.

Definition 1.1.5. A functor $F : \mathscr{C} \to \mathscr{D}$ is a tuple of two maps. One map for objects $F_{\mathscr{O}} : \mathscr{O}_{\mathscr{C}} \to \mathscr{O}_{\mathscr{D}}$ and one map for morphisms $F_{\mathscr{M}} : \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{D}}$, such that they satisfy following conditions:

(1) *Respecting Identity*: $id_{\mathscr{D}} F_{\mathscr{O}} = F_{\mathscr{M}} \mathrm{id}_{\mathscr{C}}$.

(2) *Respecting Source/Target*: $s_{\mathscr{D}} F_{\mathscr{M}} = F_{\mathscr{O}} s_{\mathscr{C}}$ and $t_{\mathscr{D}} F_{\mathscr{M}} = F_{\mathscr{O}} t_{\mathscr{C}}$.

(3) *Respecting Composition*: $F_{\mathscr{M}} m_{\mathscr{C}} = m_{\mathscr{D}}(F_{\mathscr{M}} \times F_{\mathscr{M}})$.

Example 1.1.6. The definition above allows us to define the category $\mathscr{C}$at which has objects categories and morphisms functors.

Repeating the philosophy of categories for functors leads us to the definition of a *natural transformation*.

Definition 1.1.7. Let $F, G : \mathscr{C} \to \mathscr{D}$ be two functors. A natural transformation $\alpha : F \Rightarrow G$ is a collection of maps

$$\alpha_c : F(c) \to G(c)$$

for every object $c \in \mathscr{C}$ such that for every map $f : c \to d$ the diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"'] & F(d) \arrow[d, "\alpha_d"] \\ G(c) \arrow[r, "G(f)"] & G(d) \end{tikzcd}$$

commutes.

Using natural transformations we can even build more categories.

Theorem 1.1.8. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. The collection of functors from $\mathscr{C}$ to $\mathscr{D}$, denoted by $Fun(\mathscr{C}, \mathscr{D})$ is a category with objects functors and morphisms natural transformations.*

Notation 1.1.9. For two functors $F, G : \mathscr{C} \to \mathscr{D}$, we denote the hom set in this category as $Nat(F, G)$.

This finally leads to the famous *Yoneda lemma*, which is one of the most powerful results in category theory.

Definition 1.1.10. Let $c \in \mathscr{C}$ be an object. There is a functor $\mathscr{Y}_c : \mathscr{C} \to \mathcal{S}et$ that send each object d to the set $Hom_{\mathscr{C}}(c, d)$. Functoriality follows from composition.

Lemma 1.1.11. *Let $F : \mathscr{C} \to \mathcal{S}et$ be a functor. For each object $c \in \mathscr{C}$, there is a bijection of sets*

$$Nat(\mathscr{Y}_c, F) \cong F(c)$$

induced by the map that sends each natural transformation α to the value at the identity $\alpha_c(id_c)$.

The definitions given up to here are quite cumbersome and necessitate the reader to keep track of a lot of different information. It would be helpful if we could package that same information and present it in a more elegant manner. The way we can achieve this goal is by using *simplicial sets*.

1.1.2 Simplicial Sets: A Second Look at Categories

Simplicial sets are a very powerful tool that can help us study categories.

Definition 1.1.12. Let Δ be the category with objects all non-empty finite linearly ordered sets

$$[0] = \{0\},\ [1] = \{0 \leqslant 1\},\ [2] = \{0 \leqslant 1 \leqslant 2\},\ ...$$

and morphisms order-preserving maps of linearly ordered sets.

Notation 1.1.13. There are some specific morphisms in the category Δ that we will need later on.

- For each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ there is a unique injective map

  $$d_i : [n] \to [n+1]$$

  such that $i \in [n+1]$ is not in the image. More explicitly $d_i(k) = k$ if $k < i$ and $d_i(k) = k+1$ if $k \geqslant i$.

- For each $n \geqslant 1$ and $0 \leqslant i \leqslant n$ there is a unique surjective map

  $$s_i : [n] \to [n-1]$$

  defined as follows. $s_i(k) = k$ if $k \leqslant i$ and $s_i(k) = k-1$ if $k > i$. Notice in particular that $s_i(i) = s_i(i+1) = i$ and that s_i is injective for all other values.

We have following amazing fact regarding these two classes of maps.

Remark 1.1.14. Every morphisms in Δ can be written as a finite composition of these two classes of maps stated above. The maps satisfy certain relations that can be found in [GJ09, Page 4].

Notation 1.1.15. Because of this remark we can depict the category Δ as the following

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & {[1]} \arrow[l, "s_0"'] \arrow[r, shift left=4, "d_0"] \arrow[r] \arrow[r, shift right=4, "d_2"'] & {[2]} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \end{tikzcd}$$

Having studied Δ we can finally define a simplicial set.

Definition 1.1.16. A simplicial set is a functor $X : \Delta^{op} \to \mathcal{S}\mathrm{et}$.

Remark 1.1.17. Recall that Δ^{op} is the opposite category of Δ. It has the same objects but every morphism has reverse source and targets.

Remark 1.1.18. Concretely a simplicial set is a choice of sets $X_0, X_1, X_2, ...$ which have the appropriate functions between them. Using the diagram above, we can depict a simplicial set as:

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \underset{d_2}{\overset{d_0}{\leftleftarrows\!\!\leftarrow}} X_2 \leftleftarrows\!\!\leftleftarrows \cdots$$

notice that all arrows are reversed because this functor is mapping out of the opposite category of Δ.

Definition 1.1.19. A simplicial set is a functor and so the collection of simplicial sets is itself a category with morphisms being natural transformations. We will denote this category by $s\mathcal{S}\text{et}$.

A simplicial set is an amazing object of study. In the coming two sections we will see how, depending on which aspects we focus on, a simplicial set can have a very interesting and diverse behavior. For now we focus on the categorical aspects of simplicial sets.

First we show how we can build a simplicial set out of a category.

Construction 1.1.20. Let $\mathscr{C} = (\mathscr{O}, \mathscr{M})$ be a category. Then we define $N\mathscr{C}$ as the following simplicial set. First we define it level-wise as

$$N\mathscr{C}_0 = \mathscr{O}$$

$$N\mathscr{C}_n = \mathscr{M} \underset{\mathscr{O}}{\times} \ldots \underset{\mathscr{O}}{\times} \mathscr{M}$$

where there are n factors of $\mathscr{M}$ and $n \geqslant 1$. So, the 0 level is the set of objects and at level n we have the set of n composable morphisms.

Now we construct the maps between them. It suffices to specify the maps s_i and d_i. If $n = 0$, then $s_0 : N\mathscr{C}_0 \to N\mathscr{C}_1$ is defined as $s_0 = id_{\mathscr{C}}$. Moreover, $d_0, d_1 : N\mathscr{C}_1 \to N\mathscr{C}_0$ are defined as $d_0 = s, d_1 = t$.

Let $n \geqslant 1$ and let $(f_1, f_2, \ldots, f_n) \in N\mathscr{C}_n$ be an element. For $0 \leqslant i \leqslant n+1$, we define $d_i : N\mathscr{C}_n \to N\mathscr{C}_{n-1}$ for the following 3 cases:

(i=0) $d_i((f_1, f_2, \ldots, f_n)) = (f_2, f_3, \ldots, f_n)$

$(1 \leqslant i \leqslant n)$ $d_i((f_1, f_2, \ldots, f_{i-1}, f_i, \ldots, f_n)) = (f_1, f_2, \ldots, f_{i-1}f_i, \ldots, f_n)$

(i=n+1) $d_i((f_1, f_2, \ldots, f_n)) = (f_1, f_2, \ldots, f_{n-1})$

Similarly, for $0 \leqslant i \leqslant n$ we define $s_i : \mathscr{C}_n \to \mathscr{C}_{n+1}$ for the following two cases:

$(0 \leqslant i \leqslant n)$ $s_i((f_1, f_2, \ldots, f_i, \ldots, f_n)) = (f_1, f_2, \ldots, id_{s(f_i)}, f_i, \ldots, f_n)$

(i=n+1) $s_i((f_1, f_2, \ldots, f_i, \ldots, f_n)) = (f_1, f_2, \ldots, f_i, \ldots, f_n, id_{t(f_n)})$

It is an exercise in diagram chasing to show that $N\mathscr{C}$ satisfies the relations of a simplicial set with the d_i and s_i defined above.

Remark 1.1.21. Notice in order to define $N\mathscr{C}$ it did not suffice to have a two sets with 3 maps between them. We needed the existence of the composition map to be able to make the definition work.

This construction merits a new definition.

Definition 1.1.22. Let $\mathscr{C}$ be a category. The *nerve* of $\mathscr{C}$ is the simplicial set $N\mathscr{C}$ described above.

The nerve construction fits well into our philosophy of category theory.

Theorem 1.1.23. *The nerve construction is functorial. Thus we get a functor*

$$N : \mathcal{C}at \to s\mathcal{S}et$$

Proof. We already constructed the map on objects. For a functor $F : \mathscr{C} \to \mathscr{D}$, the simplicial map $NF : N\mathscr{C} \to N\mathscr{D}$ can be defined level-wise as

- $NF_0 = F_{\mathscr{O}}$
- $NF_n = F_{\mathscr{M}} \underset{F_{\mathscr{O}}}{\times} \dots \underset{F_{\mathscr{O}}}{\times} F_{\mathscr{M}}$.

From here on it is a diagram chasing exercise to see that NF_n make all the necessary squares commute.

Note that it clearly follows that if $I_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is the identity functor, then $NI_{\mathscr{C}}$ is the identity map. Moreover, $N(F \circ G) = NF \circ NG$. □

Example 1.1.24. We have already introduced the linearly ordered set $[n]$ before (Definition 1.1.12). We can think of $[n]$ as a category, where the objects are the elements and a morphism are ordered 2-tuples (i, j), where $i \leqslant j$. The source of such map (i, j) is i and the target is j. The identity map of an element i is the tuple (i,i). Finally, we can compose two morphisms (i, j) and (j,k) to the morphism (i,k). This gives us a category, which we will still denote by $[n]$. Notice in this case for each chosen objects i, j there either is a unique morphism from i to j (if $i \leqslant j$) or there is no morphism at all.

There is a more direct way to think about the set of morphisms. The ordered set $[1]$ has two ordered elements $0 \leqslant 1$. Given that a morphism is a choice of two ordered elements, we can think of a morphism as an order preserving map $[1] \to [n]$. But that is exactly a morphism in the category Δ. Thus the set of morphisms also corresponds to $Hom_{\Delta}([1], [n])$. Let us compute $N([n])$. By definition $N([n])_0 = [n]$. Moreover, $N([n])_1 = Hom_{\Delta}([1], [n])$. Next notice that $N([n])_m = N([n])_1 \times_{N([n])_0} \dots \times_{N([n])_0} N([n])_1$, which corresponds to a choice of m ordered numbers $(i_1, i_2, \dots, i_m)$. Using the same argument as the last paragraph, we see that $N([n])_m = Hom_{\Delta}([m], [n])$. Thus, $N([n])$ is really just the representable functor

$$N([n]) = Hom_{\Delta}(-, [n]) : \Delta^{op} \to \mathcal{S}\text{et}$$

.

This simplicial set is really special and thus deserves its own name.

Definition 1.1.25. For each n there is a representable functor, which maps $[i]$ to $Hom_{\Delta}([i],[n])$. We will denote this simplicial set by $\Delta[n]$. By the Yoneda lemma, for any simplicial set X we have following isomorphism of sets:

$$Hom_{s\mathcal{S}\mathrm{et}}(\Delta[l],X) \cong X_n.$$

By now we have shown that we can take a category and build a simplicial set out of it. But can we build every simplicial set this way? If not then which ones do we get?

Definition 1.1.26. A simplicial set X satisfies the *Segal condition* if the map

$$X_n \xrightarrow{\cong} X_1 \underset{X_0}{\times} \dots \underset{X_0}{\times} X_1$$

is a bijection for $n \geqslant 2$.

The nerve $N\mathscr{C}$ satisfies the Segal condition by its very definition. Thus not every simplicial set is equivalent to the nerve of a category. But what condition other than the Segal condition do we need?

Theorem 1.1.27. *Let X be a simplicial set that satisfies the Segal condition. Then there exists a category $\mathscr{C}$ such that X is equivalent to $N\mathscr{C}$.*

Proof. We define the category $\mathscr{C}$ as follows. It has objects $\mathscr{O}_{\mathscr{C}} = X_0$ and morphisms $\mathscr{M}_{\mathscr{C}} = X_1$. Then the source, target and identity maps are defined as $s_{\mathscr{C}} = d_1 : X_1 \to X_0$, $t_{\mathscr{C}} = d_0 : X_1 \to X_0$, $id_{\mathscr{C}} = s_0 : X_0 \to X_1$ and the product map is defined as $m_{\mathscr{C}} = d_1 : X_2 \to X_1$. Here we are using the fact that $X_2 \cong X_1 \times_{X_0} X_1$. Thus we can think of m as a map $m : \mathscr{M}_{\mathscr{C}} \times \mathscr{O}_{\mathscr{C}} \mathscr{M}_{\mathscr{C}} \to \mathscr{M}_{\mathscr{C}}$, which is exactly what we wanted. The simplicial relations show that $\mathscr{C}$ satisfies the conditions stated in Definition 1.1.1.

Finally, we have the following bijection.

$$(N\mathscr{C})_n = \mathscr{M}_{\mathscr{C}} \underset{\mathscr{O}_{\mathscr{C}}}{\times} \dots \underset{\mathscr{O}_{\mathscr{C}}}{\times} \mathscr{M}_{\mathscr{C}} = X_1 \underset{X_0}{\times} \dots \underset{X_0}{\times} X_1 \cong X_n$$

This shows that $N\mathscr{C}$ is equivalent to X and finished the proof. □

The upshot is that a simplicial set that satisfies the Segal condition has the same data as a category and so instead of keeping track of all the necessary data and maps between them it packages everything very nicely and it gives us much better control. This doesn't just hold for the categories themselves, but also carries over to functors.

Theorem 1.1.28. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. Then the functor N induces a bijection of hom sets*

$$N : Hom_{\mathcal{C}at}(\mathscr{C},\mathscr{D}) \to Hom_{s\mathcal{S}et}(N\mathscr{C},N\mathscr{D})$$

Proof. We prove the result by showing the map above has an inverse. Let $f : N\mathscr{C} \to N\mathscr{D}$ be a simplicial map. Then we define $P(f)$ as the functor that is defined on objects as f_0 and defined on morphisms as f_1. The simplicial identities then show that it satisfies the conditions of a functors. Finally, for any functor $F : \mathscr{C} \to \mathscr{D}$, the composition $PN(F) = F$ by definition. On the other hand for any simplicial map $f : N\mathscr{C} \to N\mathscr{D}$, $NP(f) = f$ as they agree at level 0 and 1 and that characterizes the map completely. □

Up until now we have shown how we can use the data of a simplicial set to study categories and recover category theory. The next goal is to show we can use the same ideas to study homotopy theory.

1.1.3 Homotopy Theory of Topological Spaces

Recall the classical definition of homotopies of topological spaces.

Definition 1.1.29. Two maps of topological spaces $f,g : X \to Y$ are called *homotopic* if there exists a map $H : X \times [0,1] \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$.

Definition 1.1.30. A map $f : X \to Y$ is called a *homotopy equivalence* if there exists a map $g : Y \to X$ such that both fg and gf are homotopic to the identity map.

A key question in the homotopy theory of spaces is to determine whether a map is an equivalence or not. However topological spaces can be quite pathological and so we often look for suitable ”replacements” i.e. equivalent spaces which have a simpler structure. One good example is a CW-complex.

Theorem 1.1.31. *For each topological space X there exists a CW-complex $\tilde{X}$ and map $\tilde{X} \to X$ that is a homotopy equivalence.*

Thus from a homotopical perspective it often suffices to study CW-complexes rather than all spaces. However, a CW-complex is built out of simplices. Thus what we really care about is how many simplices we have and how they are attached to each other. This suggests that we can study spaces from the perspective of simplicial sets.

1.1.4 Simplicial Sets: A Second Look at Spaces

Here we show how we can use simplicial sets to study the homotopy theory of topological spaces. We have already defined simplicial sets in the previous section. So, first we show how to construct a simplicial set out of any topological space.

Definition 1.1.32. Let $S(l)$ be the standard $l+1$-simplex. Concretely $S(l)$ is the convex hull of the $l+1$ points $(1,0,...,0),(0,1,...,0),...,(0,0,...,1)$ in $\mathbb{R}^{l+1}$. In particular, $S(0)$ is a point, $S(1)$ is an interval and $S(2)$ is a triangle.

Remark 1.1.33. One important fact about those simplices is that the boundary is built out of lower dimensional simplices. For example, the boundary of a line is the union of two points or the boundary of a triangle is the union of three lines. This means we have two maps $d_0, d_1 : S(0) \to S(1)$ that map to the two boundary points or we have three maps $d_0, d_1, d_2 : S(1) \to S(2)$.

On the other side, we can always collapse one boundary component to lower the dimension of our simplex. Thus there are two ways to collapse our triangle $S(2)$ to a line $S(1)$, which gives us two maps $s_0, s_1 : S(2) \to S(1)$. It turns out these maps do satisfy the covariant version of the simplicial identities, which are also called the *cosimplicial identities*. This means we can thus define a functor

$$S : \Delta \to \mathcal{T}\mathrm{op}$$

This functor can be depicted in the following diagram.

$$\begin{tikzcd}
S(0) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & S(1) \arrow[l, "s_0"' near end] \arrow[r, shift left=4, "d_0"] \arrow[r] \arrow[r, shift right=4, "d_2"'] & S(2) \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4]
\end{tikzcd}$$

Definition 1.1.34. Let X be a topological space. We define the simplicial set $S(X)$ as follows. Level-wise we define $S(X)$ as

$$S(X)_n = Hom_{\mathcal{T}\mathrm{op}}(S(n), X).$$

The functoriality of I as described in the remark above shows that this indeed gives us a simplicial set.

Thus we can build a simplicial set out of every topological space. Each level indicates how many $n+1$-simplices can be mapped into our space. However, we cannot build every kind of simplicial set this way. Rather the simplicial set we constructed is called a *Kan complex*. In order to be able to give a definition we need to gain a better understanding of simplicial sets first.

Definition 1.1.35. A simplicial set T is a subsimplicial set of S, if for any l we have $T_l \subset S_l$, and for every morphism $\alpha : [k] \to [l]$, the associated map $S(\alpha) : S_l \to S_k$ carries T_l into T_k. In particular, T inherits the same face ($d_i's$) and degeneracy ($s_j's$) maps.

Example 1.1.36. There are two important classes of sub simplicial sets of $\Delta[l]$ (Definition 1.1.25):

1. The first one is denoted by $\partial\Delta[l]$ and defined as follows: $\partial\Delta[l]_i$ is the subset of all non-surjective maps in $Hom_\Delta([i],[l])$. In particular, this implies that for $i < n$, we have $\partial\Delta[l]_i = \Delta[l]_i$ and for $i = l$ we have $\partial\Delta[l]_l = \Delta[l]_l - \{id_{[l]}\}$. Intuitively it looks like the boundary of our convex space i.e. $\Delta[l]$ with the center n-dimensional cell removed.

2. The second is denoted by $\Lambda[l]_i$ ($0 \leqslant i \leqslant l$) and consists of non-surjective maps that satisfy the following condition: $(\Lambda[n]_i)_j$ is the subset of all maps in $Hom_\Delta([j],[l])$, that satisfy following condition. If i is not in the image of the map then at least one other elements also has to be not in the image. Concretely, this means it is also a subspace of $\partial\Delta[l]$ and it excludes the face which is formed by all vertices except for i. Intuitively, this one looks like a boundary where one of the faces (the one opposing the vertex i) has been removed as well. Given the resulting shape it is very often called a "horn".

Having gone through these definitions we can finally define a Kan complex.

Definition 1.1.37. A simplicial set K is called a Kan complex if for any $l \geqslant 0$ and $0 \leqslant i \leqslant l$, the map

$$Hom_{\mathbb{S}}(\Delta[l], K) \to Hom_{\mathbb{S}}(\Lambda[l]_i, K)$$

is surjective.

Remark 1.1.38. Basically the definition is saying that following diagram lifts:

$$\begin{tikzcd} \Lambda[l]_i \arrow[r] \arrow[d] & K \\ \Delta[l] \arrow[ur, dashed] & \end{tikzcd}$$

Example 1.1.39. For every topological space X, the simplicial set SX is a Kan complex. We will not prove this fact here. It relies on the idea that a topological space has no sense of direction. Thus every path can be inverted. Concretely, for any map $\gamma : I(1) \to X$, there is a map $\gamma^{-1} : I(1) \to X$ that is defined as $\gamma^{-1}(t) = \gamma(1-t)$. Thus every element $\gamma \in S(X)_1$ has a reverse path. A similar concept applies to higher dimensional maps.

It is that idea that allows us to lift any map of the form above. For a rigorous argument see [GJ09, Chapter 1].

Example 1.1.40. Contrary to the example above $\Delta[l]$ is not a Kan complex (if $l > 0$). For example the map $\Lambda[2]_0 \to \Delta[l]$ that sends 0 to 0, 1 to 2 and 2 to 1 cannot be lifted.

The definition above is a special case of a *Kan fibration*.

Definition 1.1.41. A map of simplicial sets $f : S \twoheadrightarrow T$ is a Kan fibration if any commutative square of the form

$$\begin{tikzcd}
\Lambda[l]_i \arrow[r] \arrow[d] & S \arrow[d, two heads] \\
\Delta[l] \arrow[r] \arrow[ur, dashed] & T
\end{tikzcd}$$

lifts, where $n \geqslant 0$ and $0 \leqslant i \leqslant n$.

Remark 1.1.42. This generalizes Kan complexes as K is a Kan complex if and only if the map $K \to \Delta[0]$ is a Kan fibration. As a result, if $K \twoheadrightarrow L$ is a Kan fibration and L is Kan fibrant, then K is also Kan fibrant

Kan complexes share many characteristics with topological spaces. In particular, we can talk about equivalences and homotopies.

Definition 1.1.43. Two maps $f, g : L \to K$ between Kan complexes are called *homotopic* if there exists a map $H : L \times \Delta[1] \to K$ such that $H|_0 = f$ and $H|_1 = g$.

Remark 1.1.44. This definition can be made for any simplicial set, but it is only a equivalence relation for the case of Kan complex.

Example 1.1.45. One particular instance of this definition is when $L = \Delta[0]$. In this case we have two points $x, y : \Delta[0] \to K$. We say x and y are homotopic or *equivalent* if there is a map $\gamma : \Delta[1] \to K$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 1.1.46. A map $f : L \to K$ between Kan complexes is called an *equivalence* if there are maps $g, h : K \to L$ such that $fg : K \to K$ is homotopic to id_K and $hf : L \to L$ is homotopic to id_L.

Most importantly, in order to study equivalences of spaces it suffices to study equivalences of the analogous Kan complexes.

Lemma 1.1.47. *A map of topological spaces $f : X \to Y$ is a homotopy equivalence if and only if the map of Kan complexes $Sf : SX \to SY$ is a homotopy equivalence.*

Seeing how that result holds requires us to use much more machinery. One very efficient way is to use the language of *model categories*. A model structure can capture the homotopical data in the context of a category. Using model categories we can show that topological spaces and simplicial sets (if we focus on Kan complexes) have equivalent model structures. For a better understanding of model structures see [Hir03].

Remark 1.1.48. Kan fibrations are important in the homotopy theory of simplicial sets. That is because base change along Kan fibrations is equivalence preserving. By that we mean that in the following pullback diagram

$$\begin{tikzcd} K \underset{M}{\times} L \arrow[r, "\simeq", "g^*f"'] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner", very near start] & K \arrow[d, two heads, "g"] \\ L \arrow[r, "\simeq", "f"'] & M \end{tikzcd}$$

if f is an equivalence and g is a Kan fibration then g^*f is also an equivalence. Moreover, the pullback of a Kan fibration is also a Kan fibration. Thus we say such a pullback diagram is *homotopy invariant*.

Remark 1.1.49. The homotopy invariance of base change by a Kan fibration implies in particular that we can define a *homotopy pullback*. We say a diagram of Kan complexes

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, two heads, "g"] \\ C \arrow[r, "f"'] & D \end{tikzcd}$$

is a homotopy pullback if the induced map $A \to B \times_D C$ is a homotopy equivalence. In other words, we demand a pullback "up to homotopy" rather than a strict pullback. The fact that g is a Kan fibration implies that this definition is well-defined.

Before we move on we will focus on one particular, yet very important instance of a homotopy equivalence.

Definition 1.1.50. A Kan complex K is *contractible* if the map $K \to \Delta[0]$ is a homotopy equivalence.

Remark 1.1.51. The notion of a contractible Kan complex is central in homotopy theory. It is the homotopical analogue of uniqueness as it implies that every two points in K are equivalent. Moreover, any two paths are themselves equivalent in the suitable sense and this pattern continues.

A contractible Kan complex is again a special kind of Kan fibration.

Definition 1.1.52. We say a map $K \to L$ is a *trivial Kan fibration* if it is a Kan fibration and a weak equivalence.

Lemma 1.1.53. *A map $K \to L$ is a trivial Kan fibration if and only if it is a Kan fibration and for every map $\Delta[0] \to L$, the fiber $\Delta[0] \times_L K$ is contractible.*

Remark 1.1.54. Thus a trivial Kan fibration not only has lifts, but the space of lifts is contractible, meaning there is really only one choice of lift up to homotopy.

Having a homotopical notion of an isomorphism, namely an equivalence, we can also define the homotopical version of an injection, namely a *(−1)-truncated map*.

Definition 1.1.55. A Kan fibration $K \to L$ is (-1)-truncated if for every map $\Delta[0] \to L$, the fiber $\Delta[0] \times_L K$ is either contractible or empty.

Before we move on there is one last property of Kan complexes that we need, namely that they are Cartesian closed.

Remark 1.1.56. The category of simplicial sets is Cartesian closed. For every two simplicial sets X, Y there is a mapping simplicial set, $Map(X,Y)$ defined level-wise as

$$Map(X,Y)_n = Hom(X \times \Delta[n], Y).$$

Proposition 1.1.57. *If K is a Kan complex, then for every simplicial set X, the simplicial set $Map(X,K)$ is also a Kan complex.*

Notation 1.1.58. As we have established a well functioning homotopy theory with Kan complexes, we will henceforth exclusively use the word space to be a Kan complex.

1.1.5 Two Paths Coming Together

Until now we showed that we can think of categories as a simplicial set that satisfies the Segal condition and a topological space as a Kan complex. Thus simplicial sets have two different aspects to them.

We can either think of simplicial sets that have a notion of direction and allow us to do category theory. When we think of simplicial sets this way we denote them by $s\mathcal{S}et$ and pictorially we can depict them as:

$$\begin{tikzcd} \mathscr{C}_0 \arrow[r, "id"] & \mathscr{C}_1 \arrow[l, shift right=2, "t"'] \arrow[l, shift left=2, "s"] \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathscr{C}_2 \arrow[l, "m"] \arrow[l, shift right=4] \arrow[l, shift left=4] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift left=6] \arrow[r, shift right=6] & \cdots \arrow[l] \arrow[l, shift left=4] \arrow[l, shift right=4] \arrow[l, shift left=8] \end{tikzcd}$$

On the other side, we can think of simplicial sets that have homotopical properties. In this case we call them spaces and denote that very same category as $\mathcal{S}$. This time we depict it as:

$$\begin{tikzcd}
K_0 \arrow[d, "s_0"] \\
K_1 \arrow[u, shift left=2, "d_0"] \arrow[u, shift right=2, "d_1"'] \arrow[d, shift left] \arrow[d, shift right] \\
K_2 \arrow[u, shift left=3, "d_0"] \arrow[u] \arrow[u, shift right=3, "d_2"'] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\vdots \arrow[u, shift left=3] \arrow[u, shift left=1] \arrow[u, shift right=1] \arrow[u, shift right=3]
\end{tikzcd}$$

A higher category should generalize categories and spaces at the same time. Thus our goal is it to embed both versions of simplicial sets (categorical and homotopical) into a larger setting. We need to start with a category which can house two versions of simplicial sets in itself independent of each other so that we can give each the properties we desire and make sure one part has a categorical behavior and one part has a homotopical behavior. This point of view leads us to the study of simplicial spaces.

1.1.6 Simplicial Spaces

In this section we define and study objects that have enough room to fit two versions of simplicial sets inside of it. We will call this object a *simplicial space*, although they are also known as *bisimplicial sets*. The next subsection will justify why we have decided to use the term simplicial space.

Definition 1.1.59. We define the category of *simplicial spaces* as $Fun(\Delta^{op}, \mathcal{S})$ and denote it by $s\mathcal{S}$.

Remark 1.1.60. We have the adjunction

$$Fun(\Delta^{op} \times \Delta^{op}, Set) \cong Fun(\Delta^{op}, Fun(\Delta^{op}, Set)) = Fun(\Delta^{op}, \mathcal{S}).$$

Thus on a categorical level a simplicial space is a bisimplicial set. Therefore, we can depict it at the same time as a bisimplicial set or as a simplicial space. We an depict those two as follows:

$$\begin{array}{ccccccc}
X_{00} & \leftleftarrows & X_{10} & \leftleftarrows & X_{20} & \leftleftarrows & \cdots \\
\upuparrows & & \upuparrows & & \upuparrows & & \\
X_{01} & \leftleftarrows & X_{11} & \leftleftarrows & X_{21} & \leftleftarrows & \cdots \\
\upuparrows & & \upuparrows & & \upuparrows & & \\
X_{02} & \leftleftarrows & X_{12} & \leftleftarrows & X_{22} & \leftleftarrows & \cdots \\
\upuparrows & & \upuparrows & & \upuparrows & & \\
\vdots & & \vdots & & \vdots & & \ddots \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
X_{0\bullet} & \leftleftarrows & X_{1\bullet} & \leftleftarrows & X_{2\bullet} & \leftleftarrows & \cdots
\end{array}$$

Notice that $X_{0\bullet}, X_{1\bullet}, ...$ are themselves simplicial sets.

Remark 1.1.61. There are two ways to embed simplicial sets into simplicial spaces.

1. There is a functor
$$i_F : \Delta \times \Delta \to \Delta$$
that send $([n],[m])$ to $[n]$. This induces a functor
$$i_F^* : s\mathcal{S}\mathrm{et} \to s\mathcal{S}$$
that takes a simplicial set S to the simplicial space $i_F^*(S)$ defined as follows.
$$i_F^*(S)_{kl} = S_k$$
We call this embedding the *vertical embedding*.

2. There is a functor
$$i_\Delta : \Delta \times \Delta \to \Delta$$
that send $([n],[m])$ to $[m]$. This induces a functor
$$i_\Delta^* : s\mathcal{S}\mathrm{et} \to s\mathcal{S}$$
that takes a simplicial set S to the simplicial space $i_\Delta^*(S)$ defined as follows.
$$i_F(S)_{kl} = S_l$$
We call this embedding the *horizontal embedding*.

Given that there are two embeddings there are two ways to embed generators.

Definition 1.1.62. We define $F(n) = i_F^*(\Delta[n])$ and $\Delta[l] = i_\Delta^*(\Delta[l])$. Similarly, we define $\partial F(n) = i_F^*(\partial\Delta[n])$ and $L(n)_i = i_F^*(\Lambda[n]_i)$.

The category of simplicial spaces has many pleasant features that we will need later on.

Definition 1.1.63. The category of simplicial spaces is Cartesian closed. For any two objects X and Y we define the simplicial space Y^X as

$$(Y^X)_{nl} = Hom_{s\mathcal{S}}(F(n) \times \Delta[l] \times X, Y)$$

Remark 1.1.64. In particular, the previous statement implies that $s\mathcal{S}$ is enriched over simplicial sets, as for every X and Y, we have a mapping space $Map_{s\mathcal{S}}(X,Y) = (Y^X)_0$.

Remark 1.1.65. Using the enrichment, by the Yoneda lemma, for any simplicial space X we have following isomorphism of simplicial sets:

$$Map_{s\mathcal{S}}(F(n), X) \cong X_n.$$

Section 1.2
Complete Segal spaces

We start with $n = 1$. An $(\infty, 1)$-category should be a 1-category up to coherent homotopy which is encoded in the invertible higher morphisms. In this section, we will discuss one particular model for $(\infty, 1)$-categories. A good overview on different models for $(\infty, 1)$-categories and their comparison can be found in [Ber10]. It should be mentioned that by [To5] up to equivalence, there is essentially only one theory of $(\infty, 1)$-categories; explicit equivalences between the models mentioned here have been proved e.g. in [DKS89, Ber07, BK12, Hor15]. One additional model which should be mentioned is that of Joyal's quasi-categories. It has been intensively studied, most prominently in [Lur09a].

1.2.1 The homotopy hypothesis and $(\infty, 0)$-categories

The basic hypothesis upon which ∞-category theory is based goes back to Grothendieck [Gro21] and is the following:

Hypothesis 1.2.1 (Homotopy hypothesis)**.** Spaces are models for ∞-groupoids, also referred to as $(\infty, 0)$-categories.

Given a space X, its points, i.e. 0-simplices, are thought of as objects of the $(\infty, 0)$-category, paths between points as 1-morphisms, homotopies between paths as 2-morphisms, homotopies between homotopies as 3-morphisms, and so forth. With this interpretation, it is clear that all n-morphisms are invertible up to homotopies, which are higher morphisms.

We take this hypothesis as the basic definition, and model "spaces" with simplicial sets rather than with topological spaces.

Definition 1.2.2. An $(\infty, 0)$-*category*, or ∞-*groupoid*, is a space. According to our conventions, it is a fibrant simplicial set, i.e. a Kan complex.

1.2.2 Topologically and simplicially enriched categories

Two particularly simple, but quite rigid models are topologically or simplicially enriched categories.

Definition 1.2.3. A *topological category* is a category enriched in topological spaces. A *simplicial category* is a category enriched in simplicial sets.

Topological and simplicial categories are discussed and used in [Lur09a, TV05]. These models of $(\infty,1)$-categories are perhaps the easiest to visualize and are a great psychological aid but are rigid to work with in practice because, among other problems, enriched functors do not furnish all homotopy classes of functors between the $(\infty,1)$-categories being modeled, unless the domain and codomain satisfy appropriate conditions[a]. And even when those conditions are met, the category of enriched functors might not correctly model the $(\infty,1)$-functor category, essentially because enriched functors correspond to functors that preserve composition strictly while $(\infty,1)$-functors are allowed to preserve it only up to coherent homotopy. For our applications, we would also like to allow some flexibility for objects, not only morphisms, thus also requiring spaces of objects.

1.2.3 Segal spaces

Complete Segal spaces, first introduced by Rezk in [Rez01] as a model for $(\infty,1)$-categories, turn out to be very well-suited for geometric applications. We recall the definition in this section.

Definition 1.2.4. A (1-*fold*) *Segal space* is a simplicial space $X = X_\bullet$ which satisfies the *Segal condition*: for any $n,m \geqslant 0$ the commuting square

$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_n \arrow[r] & X_0 \end{tikzcd}$$

induced by the maps $[m] \to [m+n]$, $(0 < \cdots < m) \mapsto (0 < \cdots < m)$, and $[n] \to [m+n]$, $(0 < \cdots < n) \mapsto (m < \cdots < m+n)$, is a homotopy pullback square. In other words, the induced map

$$X_{m+n} \longrightarrow X_m \overset{h}{\underset{X_0}{\times}} X_n$$

is a weak equivalence.

Defining a *map of Segal spaces* to be a map of the underlying simplicial spaces gives a category of Segal spaces $\mathscr{S}e\mathscr{S}p = \mathscr{S}e\mathscr{S}p_1$.

Remark 1.2.5. For any $m \geqslant 1$, consider the maps $g_\beta : [1] \to [m]$, $(0 < 1) \mapsto (\beta - 1 < \beta)$ for $1 \leqslant \beta \leqslant m$. Note that requiring the Segal condition is equivalent to requiring the condition that the maps

$$X_m \longrightarrow X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

induced by $g_1, \ldots, g_m$ are weak equivalences.

[a]Namely, that the domain be cofibrant and the codomain be fibrant in appropriate model structures.

Remark 1.2.6. Following [Lur09c] we omit the Reedy fibrancy condition which often appears in the literature. In particular, this condition would guarantee that for $m, n \geqslant 0$ the canonical map

$$X_m \underset{X_0}{\times} X_n \longrightarrow X_m \underset{X_0}{\overset{h}{\times}} X_n$$

is a weak equivalence. Our definition corresponds to the choice of the projective model structure instead of the injective (Reedy) model structure, which is slightly different (though Quillen equivalent) compared to [Rez01]. We will explain this in more detail in Section 1.2.4.

Definition 1.2.7. We will refer to the spaces X_n as the *levels* of the Segal space X.

Example 1.2.8. Let $\mathscr{C}$ be a small topological category. Recall that its nerve is the simplicial set

$$N(\mathscr{C})_n = \mathrm{Hom}([n], \mathscr{C}) = \coprod_{x_0,\dots,x_n \in \mathrm{Ob}\,\mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(x_0, x_1) \times \cdots \mathrm{Hom}_{\mathscr{C}}(x_{n-1}, x_n),$$

with face maps given by composition of morphisms, and degeneracies by insertions of identities. The nerve $N(\mathscr{C})$ is a Segal space. Moreover, a simplicial set, viewed as a simplicial space with discrete levels, satisfies the Segal condition if and only if it is the nerve of an (ordinary) category.

Segal spaces as $(\infty, 1)$-categories

The above example motivates the following interpretation of Segal spaces as models for $(\infty, 1)$-categories. If $X_\bullet$ is a Segal space then we view the set of 0-simplices of the space X_0 as the set of objects. For $x, y \in X_0$ we view

$$\mathrm{Hom}_X(x, y) = \{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}$$

as the $(\infty, 0)$-category, i.e. the space, of arrows from x to y. More generally, we view X_n as the $(\infty, 0)$-category, i.e. the space, of n-tuples of composable arrows together with a composition. Note that given an n-tuple of composable arrows, the Segal condition implies that the corresponding fiber of the Segal map $X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ is a contractible space. The map $X_n \to X_1$ determined by the functor $[1] \to [n], 0 < 1 \mapsto 0 < n$ can be thought of as "composition", and thus we can think of the n-tuple as having a contractible space of possible compositions. Moreover, one can interpret paths in the space X_1 of 1-morphisms as 2-morphisms, which are invertible up to homotopies, which in turn are 3-morphisms, and so forth.

The homotopy category of a Segal space

To a higher category one can intuitively associate an ordinary category, its *homotopy category*, which has the same objects and whose morphisms are 2-isomorphism classes of 1-morphisms. For Segal spaces, one can realize this idea as follows.

Definition 1.2.9. The *homotopy category* $h_1(X)$ of a Segal space $X = X_\bullet$ is the (ordinary) category whose objects are the 0-simplices of the space X_0 and whose morphisms between objects $x, y \in X_0$ are

$$\mathrm{Hom}_{h_1(X)}(x,y) = \pi_0\left(\mathrm{Hom}_X(x,y)\right) = \pi_0\left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\}\right)$$

For $x, y, z \in X_0$, the following diagram induces the composition of morphisms, as weak equivalences induce bijections on π_0.

$$\begin{aligned}\left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\}\right) \times \left(\{y\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{z\}\right) \quad &\longrightarrow \quad \{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{z\} \\ &\overset{\simeq}{\longleftarrow} \quad \{x\} \overset{h}{\underset{X_0}{\times}} X_2 \overset{h}{\underset{X_0}{\times}} \{z\} \\ &\longrightarrow \quad \{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{z\}.\end{aligned}$$

Example 1.2.10. Given a small (ordinary) category $\mathscr{C}$, the homotopy category of its nerve, viewed as a simplicial space with discrete levels, is equivalent to $\mathscr{C}$,

$$h_1(N(\mathscr{C})) \simeq \mathscr{C}.$$

The above example motivates the following definition of equivalences of Segal spaces.

Definition 1.2.11. A map $f : X \to Y$ of Segal spaces is a *Dwyer-Kan equivalence* if

1. the induced map $h_1(f) : h_1(X) \to h_1(Y)$ on homotopy categories is essentially surjective, and
2. for each pair of objects $x, y \in X_0$ the induced map $\mathrm{Hom}_X(x,y) \to \mathrm{Hom}_Y(f(x), f(y))$ is a weak equivalence.

1.2.4 Complete Segal spaces

We would like the equivalences of Segal spaces to be the Dwyer-Kan equivalences. However, instead of considering all Segal spaces and their the Dwyer-Kan equivalences, it turns out that we can instead consider a full subcategory of Segal spaces which satisfy an extra condition called *completeness*, for which Dwyer-Kan equivalences have an equivalent, simpler, description. To make sense of this, we need to first introduce the model categories involved.

The model structures of Segal spaces

We now describe various model structures on the category $s\mathscr{S}pace$ of simplicial spaces in this section. Ultimately, the goal is to have a model category whose fibrant objects deserve to be called "$(\infty, 1)$-categories" and whose equivalences are analogs of equivalences of categories. We will first introduce model categories whose fibrant objects are Segal spaces. Then, in the next step, we will fix the weak equivalences. We refer to [Rez01] and [Hor15] for more details.

Let us first consider the injective and projective model structures on the category of simplicial spaces, denoted by $s\mathscr{S}pace_c$ and $s\mathscr{S}pace_f$, respectively. Note that the fibrant objects in $s\mathscr{S}pace_f$ are the levelwise fibrant ones, while the fibrant objects of $s\mathscr{S}pace_c$ turn out to be the *Reedy fibrant* simplicial spaces[b]. Conversely, every object in $s\mathscr{S}pace_c$ is cofibrant, see for example [Hir03, Corollary 15.8.8.]. These model categories are Quillen equivalent (via the identity functor).

In the first step we perform left Bousfield localizations of the previous model structures $s\mathscr{S}pace_c$ and $s\mathscr{S}pace_f$ with respect to the morphisms

$$\Delta^1 \sqcup_{\Delta^0} \cdots \sqcup_{\Delta^0} \Delta^1 \longrightarrow \Delta^n.$$

This provides two model categories, denoted $s\mathscr{S}pace_c^{Se}$ and $s\mathscr{S}pace_f^{Se}$, which still are Quillen equivalent. For the injective model structure, it is immediate that fibrant objects in $s\mathscr{S}pace_c^{Se}$ satisfy $X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ and thus are Reedy fibrant Segal spaces. For the projective model structure, it follows from [Hor15] that the fibrant objects in $s\mathscr{S}pace_f^{Se}$ satisfy $X_n \xrightarrow{\simeq} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ and thus are Segal spaces[c].

Complete Segal spaces

Even though the model categories $s\mathscr{S}pace_c^{Se}$ and $s\mathscr{S}pace_f^{Se}$ have the (Reedy fibrant) Segal spaces as their fibrant objects, there are not enough weak equivalences: every weak equivalence between Segal spaces is indeed a Dwyer-Kan equivalence, but there are more Dwyer-Kan equivalences.

This problem can be circumvented by further localizing the model structures. For this new model structure, the weak equivalences between Segal spaces turn out to be exactly the Dwyer-Kan equivalences. We will see that these further localized model structures have fewer fibrant objects, which are the *complete* (Reedy fibrant) Segal spaces. We will focus on the case of the projective model structure, since the other case can be found spelled out in great detail in many references, for example the original [Rez01], but to our knowledge the former has so far only appeared in [Hor15].

[b] See for example [Hir03, Theorem 15.8.7] for a proof that the injective and Reedy model structures coincide.

[c] Note that this terminology is not consistent throughout the literature: often "Segal space" includes the Reedy fibrancy condition. Our examples will not be Reedy fibrant, which is the reason for our choice of terminology.

Moreover, although we will phrase it for the projective model structure, the first part works the same in the injective case. The difference appears when computing the involved mapping spaces explicitly, see the remark below.

Intuitively, the condition we would like to impose is that the underlying ∞-groupoid of invertible morphisms of the Segal space $X_\bullet$ is already encoded by the space X_0. To translate this, we first need to understand what the space of (homotopy) invertible morphisms of $X_\bullet$ is.

Let f be an element in X_1 with source and target x and y, i.e. its images under the two face maps $X_1 \rightrightarrows X_0$ are x and y. It is called *invertible* if its image under

$$\{x\} \underset{X_0}{\times} X_1 \underset{X_0}{\times} \{y\} \longrightarrow \{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\} \longrightarrow \pi_0\left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\}\right) = \mathrm{Hom}_{\mathrm{h}_1(X)}(x,y)\,,$$

is an invertible morphism in $h_1(X)$, i.e. it has a left and right inverse.

To define the space of invertible morphisms, consider the walking isomorphism $I[1]$, which is the category with two objects and one invertible morphism between them,

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \bullet \arrow[l, bend left] \end{tikzcd}$$

Mapping the walking isomorphism into an arbitrary category $\mathscr{C}$ we get the isomorphisms of $\mathscr{C}$, and therefore the information about its underlying groupoid. Mimicking this procedure for a Segal space $X_\bullet$, we consider the derived mapping space

$$\mathrm{Map}_{s\mathscr{S}pace^{Se}_f}(N(I[1]),X).$$

Moreover, an analog of [Rez01, Lemma 5.8] shows that if an element in X_1 is invertible, any element in the same connected component will also be invertible. Thus we define the *space of invertible morphisms in* $X_\bullet$ to be the homotopy pullback[d]

$$\begin{tikzcd} X_1^{inv} \arrow[rr] \arrow[d] \arrow[dr, phantom, "h\lrcorner", very near start] & & X_1 \arrow[d] \\ \pi_0\mathrm{Map}_{s\mathscr{S}pace^{Se}_f}(N(I[1]),X) \arrow[r] & \pi_0 X_1 = \pi_0\mathrm{Map}_{s\mathscr{S}pace^{Se}_f}(\Delta^1,X) & \end{tikzcd}$$

Here, the bottom arrow arises from the obvious functor $[1] \to I[1]$.

Finally, identity morphisms in $X_\bullet$ should be invertible. Indeed, the degeneracy map $s_0 : [1] \to [0]$ factors as $[1] \to I[1] \to [0]$ and induces a map

$$X_0 \to X_1^{inv}.$$

[d]To compare with the definition in [Hor15], note that the pullback is a homotopy pullback since the map $X_1 \to \pi_0(X_1)$ is a fibration.

Definition 1.2.12. A Segal space $X_\bullet$ is *complete* if the map $X_0 \to X_1^{inv}$ is a weak equivalence. We denote the full subcategory of $\mathscr{S}e\mathscr{S}p$ whose objects are complete Segal spaces by $\mathscr{C}\mathscr{S}\mathscr{S}p = \mathscr{C}\mathscr{S}\mathscr{S}p_1$.

Example 1.2.13. Let $\mathscr{C}$ be a category. Then $N(\mathscr{C})$ is a complete Segal space if and only if there are no non-identity isomorphisms in $\mathscr{C}$, i.e. the underlying groupoid of $\mathscr{C}$ is a set (viewed as a category with only identity morphisms).

In order to compute X_1^{inv} explicitly, we have to be able to describe the (derived) mapping space $\mathrm{Map}_{s\mathscr{S}pace_f^{Se}}(N(I[1]), X)$.

Lemma 1.2.14. *We have a homotopy pullback square*

$$\begin{tikzcd}
\mathrm{Map}_{s\mathscr{S}pace_f^{Se}}(N(I[1]),X) \arrow[r] \arrow[d, "\{0,2\}\sqcup\{1,3\}"'] \arrow[dr, phantom, "h\lrcorner"] & X_3 \arrow[d, "\{0,2\}\sqcup\{1,3\}"] \\
X_0 \times X_0 \arrow[r] & X_1 \times X_1 .
\end{tikzcd}$$

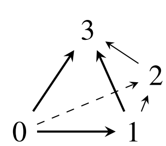
Proof. Note that since $X_\bullet$ was assumed to be a Segal space, it is fibrant, but $N(I[1])$ might not be cofibrant[e]. So to compute the desired mapping space, we cofibrantly replace $N(I[1])$ and then compute the mapping space in the underlying category,

$$\mathrm{Map}_{s\mathscr{S}pace_f^{Se}}(N(I[1]),X) \simeq \mathrm{Map}_{s\mathscr{S}pace}(\mathrm{cof}(N(I[1])),X).$$

To compute the cofibrant replacement, the crucial observation (originally by [Rez01], reformulated by [BSP21]) is that the nerve of $I[1]$ can be obtained by the pushout of simplicial sets

$$K = \Delta^3 \sqcup_{\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}}} (\Delta^0 \sqcup \Delta^0).$$

This can be seen as contracting the edges $\{0,2\}$ and $\{1,3\}$ in the 3-simplex:



We use an argument similar to that in [JFS17, Remark 3.4], which observes the following: K is given by a strict pushout along a diagram of cofibrant objects of which one arrow is an inclusion. By [Lur09a, A.2.4.4], this is a homotopy pushout in the injective model structure and therefore

[e]Note that for the injective model structure, it is cofibrant and therefore X_1^{inv} is just the subspace of X_1 of invertible morphisms.

homotopy equivalent to the homotopy pushout in the projective model structure. So a cofibrant replacement of K is given by taking the homotopy pushout of the same diagram,

$$\operatorname{cof}(K) = \Delta^3 \sqcup^h_{\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}}} (\Delta^0 \sqcup \Delta^0).$$

Finally, we obtain the space as the wanted homotopy pullback[f] □

Complete Segal spaces as fibrant objects

There is a further model structure on the category of simplicial spaces which implements completeness. It is obtained by a further left Bousfield localization, with respect to the morphism

$$\Delta^0 \longrightarrow N(I[1]).$$

This provides two Quillen equivalent model categories, denoted $s\mathscr{S}pace_c^{CSe}$ and $s\mathscr{S}pace_f^{CSe}$. Fibrant objects in $s\mathscr{S}pace_c^{CSe}$, respectively $s\mathscr{S}pace_f^{CSe}$, are Reedy fibrant complete Segal spaces, respectively complete Segal spaces.

Summarizing, we have the following diagram

$$\begin{tikzcd}
s\mathscr{S}pace_c \arrow[r, shift right] & s\mathscr{S}pace_f \arrow[l, shift right] \\
s\mathscr{S}pace_c^{Se} \arrow[u] \arrow[r, shift right] & s\mathscr{S}pace_f^{Se} \arrow[u] \arrow[l, shift right] \\
s\mathscr{S}pace_c^{CSe} \arrow[u] \arrow[r, shift right] & s\mathscr{S}pace_f^{CSe} \arrow[u] \arrow[l, shift right]
\end{tikzcd}$$

where the horizontal arrows are Quillen equivalences induced by the identity and the vertical arrows are localizations.

The following Proposition shows that in the localized model structure Dwyer-Kan equivalences of Segal spaces indeed are weak equivalence, and therefore we have fixed the concern mentioned above. We refer to [Hor15, Theorem 5.15] for a proof, which makes substantial use of the analogous result for Reedy fibrant Segal spaces in $s\mathscr{S}pace_c^{CSe}$ from [Rez01, Theorem 7.7].

Theorem 1.2.15. *Let X and Y be Segal spaces. A morphism $f : X \to Y$ is a weak equivalence in $s\mathscr{S}pace_f^{CSe}$ if and only if it is a Dwyer-Kan equivalence.*

[f]This can be compared to Rezk's definition using the zig-zag category $0 \to 2 \leftarrow 1 \to 3$ and requiring the morphisms $0 \to 2$ and $1 \to 3$ to be identities.

As a consequence the obvious inclusions induce the following equivalences of categories:

$$\mathscr{CSS}p[\mathscr{lwe}^{-1}] \longrightarrow \mathscr{S}e\mathscr{S}p[\mathscr{DK}^{-1}] \longrightarrow Ho(s\mathscr{S}pace_f^{CSe}),$$

where $\mathscr{DK}$ and $\mathscr{lwe}$ stand for the subcategory of Dwyer-Kan and levelwise weak equivalences, respectively.

This justifies the following definition.

Definition 1.2.16. An $(\infty, 1)$-*category* is a complete Segal space.

Remark 1.2.17. We denote the category of Reedy fibrant complete Segal spaces by $\mathscr{CSS}p_c$, that is to say the fibrant objects in $s\mathscr{S}pace_c^{CSe}$. Remember that $s\mathscr{S}pace_c^{CSe}$ and $s\mathscr{S}pace_f^{CSe}$ are Quillen equivalent, so that the embedding $\mathscr{CSS}p_c \subset \mathscr{CSS}p$ induces an equivalence $\mathscr{CSS}p_c[\mathscr{lwe}^{-1}] \to \mathscr{CSS}p[\mathscr{lwe}^{-1}]$ of which an inverse is given by the Reedy fibrant replacement functor $(-)^R$. Sometimes it turns out to be more useful to work in the model category $s\mathscr{S}pace_c^{CSe}$ as every object is cofibrant. Note that the Reedy fibrant replacement functor does not change the homotopy type of the levels.

Definition 1.2.18. The fibrant replacement functor in the model category $s\mathscr{S}pace_f^{CSe}$ sending a Segal space to its fibrant replacement is called *completion*. In [Rez01] Rezk gave a rather explicit construction of the completion of Segal spaces. He showed that there is a completion functor which to every Segal space X associates a complete Segal space $\widehat{X}$ together with a map $i_X : X \to \widehat{X}$, which is a Dwyer-Kan equivalence.

Remark 1.2.19. The completeness condition says that all invertible morphisms essentially are just identities up to the choice of a path. In this sense, one might like to think of complete Segal spaces as a homotopical version of skeletal[g] or *reduced* category, and, since any category is equivalent to a reduced one, assuming this extra condition is harmless. However, the information on the invertible morphisms is merely encoded in a different way, namely, in the spatial structure. Also, we would like to remark that in the homotopical situation, this intuition might be misleading: indeed, instead of thinking of a complete Segal space as having few invertible morphisms, it is better to think of a complete Segal space as having a "maximal" space of objects. This is illustrated by [Rez01, Corollary 6.6]. A good example to keep in mind is a special case of [Rez01, Remark 14.1]: given a group G, we can view as a category with one object, and consider its nerve. Its completion is the constant simplicial space BG.

Remark 1.2.20. It is worth noticing that $s\mathscr{S}pace_f$, $s\mathscr{S}pace_c$, $s\mathscr{S}pace_f^{Se}$, $s\mathscr{S}pace_c^{Se}$, $s\mathscr{S}pace_f^{CSe}$, and $s\mathscr{S}pace_c^{CSe}$ are all Cartesian closed simplicial model categories. In particular, for any simplicial space X and any complete Segal space Y, the simplicial space Y^X is a complete Segal space.

[g]A category is called *skeletal* if each isomorphism class contains just one element, see for example [Rie17].

Section 1.3
Complete n-fold Segal spaces

As a model for (∞,n)-categories, we will use complete n-fold Segal spaces, which were first introduced by Barwick in his thesis and appeared prominently in Lurie's [Lur09c]. Details can be found e.g. in [Lur09b, BSP21, BR13]. (∞,n)-categories are homotopical versions of weak n-categories. Recall that n-categories are inductively built by taking categories (weakly) enriched in $(n-1)$-categories. For $n=2$ these are known as 2-categories (strict) or bicategories (weak). Alternatively, one could choose to consider categories *internal* to $(n-1)$-categories, i.e. they have a whole $(n-1)$-category of objects. For $n=2$ these were first introduced under the name of double categories by Ehresmann in [Ehr63] and have been thoroughly studied in category theory. Therefore we will call the higher versions thereof *n-uple categories*[h]. Even though we present our main example as an n-fold Segal space in the next part, it actually arises from such an "n-uple" version as we will see later on.

Moreover, it even comes from a more rigid model, namely from internal n-uple categories, which are n-uple categories internal to simplicial sets. This model is the easiest to define, which is why we start with it.

1.3.1 Internal n-uple categories

Iterating the approach in [Hor15], one obtains a model for (∞,n)-uple-categories given by n-uple categories internal to simplicial sets, i.e. categories internal to the category of $(n-1)$-uple categories internal to simplicial sets. Unravelling the definition for $n=2$, there is a space of objects, a space of "horizontal" 1-morphisms, a space of "vertical" 1-morphisms, and a space of 2-morphisms, together with unit maps and composition maps. For larger n, there is a space of objects and suitable spaces of higher morphisms "in all directions", again together with unit maps and composition maps. Equivalently, an n-uple category internal to simplicial sets is a simplicial object in (strict) n-fold categories. This model has been discussed in [CH16].

Our bordism category defined in the next part secretly is such an internal n-uple category, however, details on this model were not available at the time of writing this article, so we will present it in a different way here.

Remark 1.3.1. Note that composition is well-defined on the nose, as opposed to the models we will consider in the next sections.

[h]This is non-standard: usually they are called n-fold categories. However, by an unfortunate choice of terminology, complete n-fold Segal spaces will correspond to n-categories. In order to hopefully reduce confusion we will instead be consistent in using "uple" for internal versions and reserve "fold" for the enriched, globular version.

1.3.2 n-uple and n-fold Segal spaces

Recall that an n-uple[i] simplicial space is a functor $X : (\Delta^{op})^{\times n} \to \mathscr{S}pace$. An n-uple Segal space is an n-uple simplicial space with an extra condition ensuring it is the ∞-analog of an n-uple category.

Definition 1.3.2. An *n-uple Segal space* is an n-uple simplicial space $X = X_{\bullet,\dots,\bullet}$ such that for every $1 \leqslant i \leqslant n$, and every $k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \geqslant 0$,

$$X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

is a Segal space.

Defining a *map of n-uple Segal spaces* to be a map of the underlying n-uple simplicial spaces gives a category of n-uple Segal spaces, $\mathscr{S}e\mathscr{S}p^n$.

Imposing an extra globularity condition leads to a model for ∞-analogs of n-categories:

Definition 1.3.3. An n-uple simplicial space $X_{\bullet,\dots,\bullet}$ is *essentially constant* if the map from the constant n-uple simplicial space $X_{0,\dots,0}$ given by the degeneracy maps

$$X_{0,\dots,0} \longrightarrow X$$

is a weak equivalence of n-uple simplicial spaces.

Definition 1.3.4. An *n-fold Segal space* is an n-uple Segal space $X = X_{\bullet,\dots,\bullet}$ such that for every $1 \leqslant i \leqslant n$, and every $k_1, \dots, k_{i-1} \geqslant 0$, the $(n-i)$-uple simplicial space

$$X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

is essentially constant.[j]

Defining a *map of n-fold Segal spaces* to be a map of the underlying n-uple simplicial spaces gives a category of n-fold Segal spaces, $\mathscr{S}e\mathscr{S}p_n$.

Remark 1.3.5. Alternatively, one can formulate the conditions iteratively. First, an *n-uple Segal space* is a simplicial object $Y_\bullet$ in $(n-1)$-uple Segal spaces which satisfies the Segal condition. Then, an n-fold Segal space is a simplicial object $Y_\bullet$ in $(n-1)$-fold Segal spaces which satisfies the Segal condition and such that Y_0 is essentially constant (as an $(n-1)$-fold Segal space). To get back the above definition, the ordering of the indices is crucial: $X_{k_1,\dots,k_n} = (Y_{k_1})_{k_2,\dots,k_n}$.

[i]Again, usually, this is called an n-fold simplicial space, but we use this terminology to emphasize the difference.

[j]To be consistent with our choice of "uple" versus "fold", we could call an n-uple simplicial space which satisfies this extra condition an n-fold simplicial space.

Interpretation as higher categories

An n-fold Segal space can be thought of as a higher category in the following way.

The first condition means that this is an n-uple category, i.e. there are n different "directions" in which we can "compose". An element of $X_{k_1,\dots,k_n}$ should be thought of as a composition consisting of k_i composable morphisms in the ith direction.

The second condition imposes that we indeed have a higher n-category, i.e. an n-morphism has as source and target two $(n-1)$-morphisms which themselves have the "same" (in the sense that they are homotopic) source and target.

For $n = 2$ one can think of this second condition as "fattening" the objects in a bicategory. A 2-morphism in a bicategory can be depicted as

$$\begin{tikzcd}
\bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet
\end{tikzcd}$$

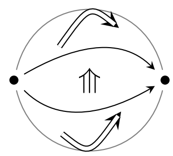
The top and bottom arrows are the source and target, which are 1-morphisms between the same objects.

In a 2-fold Segal space $X_{\bullet,\bullet}$, an element in $X_{1,1}$ can be depicted as

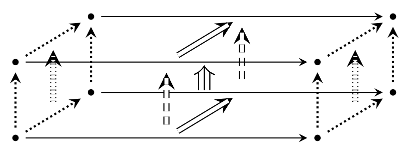
$$\begin{tikzcd}
X_{0,0} \ni & & & \in X_{0,0} \\
& \bullet \arrow[r, "X_{1,0} \ni"] \arrow[d, dotted, "X_{0,0} \simeq X_{0,1} \ni"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, dotted, "\in X_{0,1} \simeq X_{0,0}"] & \\
& \bullet \arrow[r, "\in X_{1,0}"'] & \bullet & \\
X_{0,0} \ni & & & \in X_{0,0}
\end{tikzcd}$$

The images under the source and target maps in the first direction $X_{1,1} \rightrightarrows X_{1,0}$ are 1-morphisms which are depicted by the horizontal arrows. The images under the source and target maps in the second direction $X_{1,1} \rightrightarrows X_{0,1}$ are 1-morphisms, depicted by the dashed vertical arrows, which are essentially just identity maps, up to homotopy, since $X_{0,1} \simeq X_{0,0}$. Thus, here the source and target 1-morphisms (the horizontal ones) themselves do not have the same source and target anymore, but up to homotopy they do.

The same idea works with higher morphisms, in particular one can imagine the corresponding diagrams for $n = 3$. A 3-morphism in a tricategory can be depicted as



whereas a 3-morphism, i.e. an element in $X_{1,1,1}$ in a 3-fold Segal space X can be depicted as



Here the dotted arrows are those in $X_{0,1,1} \simeq X_{0,0,1} \simeq X_{0,0,0}$ and the dashed ones are those in $X_{1,0,1} \simeq X_{1,0,0}$.

Thus, we should think of the set of 0-simplices of the space $X_{0,\dots,0}$ as the objects of our category, and elements of $X_{1,\dots,1,0,\dots,0}$ as i-morphisms, where $0 < i \leqslant n$ is the number of 1's. Pictorially, they are the i-th "horizontal" arrows. Moreover, the other "vertical" arrows are essentially just identities of lower morphisms. Similarly to before, paths in $X_{1,\dots,1}$ should be thought of as $(n+1)$-morphisms, which therefore are invertible up to a homotopy, which itself is an $(n+2)$-morphism, and so forth.

The homotopy bicategory of a 2-fold Segal space

To any higher category one can intuitively associate a bicategory having the same objects and 1-morphisms, and with 2-morphisms being 3-isomorphism classes of the original 2-morphisms.

Definition 1.3.6. The *homotopy bicategory* $\mathrm{h}_2(X)$ of a 2-fold Segal space $X = X_{\bullet,\bullet}$ is defined as follows: objects are the points of the space $X_{0,0}$ and

$$\mathrm{Hom}_{\mathrm{h}_2(X)}(x,y) = \mathrm{h}_1\big(\mathrm{Hom}_X(x,y)\big) = \mathrm{h}_1\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right)$$

as Hom categories. Horizontal composition is defined as follows:

$$\begin{aligned}
\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right) \times \left(\{y\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}\right) &\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\
&\overset{\sim}{\longleftarrow} \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{2,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\
&\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}.
\end{aligned}$$

The second arrow happens to go in the wrong way but it is a weak equivalence. Therefore after taking h_1 it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).

A proof that this definition indeed gives a bicategory will be the subject of a subsequent article.

1.3.3 Complete and hybrid n-fold Segal spaces

As with (1-fold) Segal spaces, we need to impose an extra condition to ensure that invertible k-morphisms are paths in the space of $(k-1)$-morphisms. Again, there are several ways to include its information.

Definition 1.3.7. Let X be an n-fold Segal space and $1 \leqslant i, j \leqslant n$. It is said to satisfy

CSS^i if for every $k_1, \ldots, k_{i-1} \geqslant 0$,

$$X_{k_1,\ldots,k_{i-1},\bullet,0,\ldots,0}$$

is a complete Segal space.

SC^j if for every $k_1, \ldots, k_{j-1} \geqslant 0$,

$$X_{k_1,\ldots,k_{j-1},0,\bullet,\ldots,\bullet}$$

is discrete, i.e. a discrete space viewed as a constant $(n-j+1)$-fold Segal space.

Definition 1.3.8. An n-fold Segal space is

1. *complete*, if for every $1 \leqslant i \leqslant n$, X satisfies (1.3.7).
2. a *Segal n-category* if for every $1 \leqslant j \leqslant n$, X satisfies (1.3.7).
3. *m-hybrid* for $m \geqslant 0$ if condition (1.3.7) is satisfied for $i > m$ and condition (1.3.7) is satisfied for $j \leqslant m$.

Denote the full subcategory of $\mathscr{S}e\mathscr{S}p_n$ of complete n-fold Segal spaces by $\mathscr{C}\mathscr{S}\mathscr{S}p_n$.

Remark 1.3.9. Note that an n-hybrid n-fold Segal space is a Segal n-category, while an n-fold Segal space is 0-hybrid if and only if it is complete.

For our purposes, the model of complete n-fold Segal spaces is well-suited, which leeds us to the following definition.

Definition 1.3.10. An (∞, n)-category is a complete n-fold Segal space.

The underlying model categories

Similarly to subsection 1.2.4 there are model categories running in the background. We can consider either the injective or projective model structure on the category of n-uple simplicial spaces $s\mathscr{S}pace^n$, which we denote by $s\mathscr{S}pace^n_c$ respectively $s\mathscr{S}pace^n_f$. Bousfield localizations at the analogs of the Segal maps give model structures whose fibrant objects are (Reedy fibrant) n-uple Segal spaces, further localizing at maps governing essential constancy, the fibrant objects become (Reedy fibrant) n-fold Segal spaces, and a third localization at a map imposing completeness gives model structures $s\mathscr{S}pace^{CSe}_{n,c}$ respectively $s\mathscr{S}pace^{CSe}_{n,f}$ whose fibrant objects are (Reedy fibrant) complete n-fold Segal spaces, see [Lur09b, BSP21] and [JFS17, Appendix]. Note that again, the identity map induces a Quillen equivalence between $s\mathscr{S}pace^n_c$ and $s\mathscr{S}pace^n_f$ which descends to the localizations.

Alternatively, and by [JFS17, Appendix, Proposition A.9] equivalently, the construction of complete Segal objects for absolute distributors from [Lur09b] provides an iterative definition of these model categories by considering simplicial objects in a suitable model category (which is taken to be the appropriate localization of $s\mathscr{S}pace_{n-1,c}$ respectively $s\mathscr{S}pace_{n-1,f}$) and localizing at the maps governing the Segal condition, essential constancy, and/or completeness in the new simplicial direction.

[Lur09b] also provides a model category whose fibrant objects are Segal category objects in some suitable underlying model category, thus allowing an iteration of the construction of Segal categories as well. Applying this construction m times to the above one for complete $(n-m)$-fold Segal spaces provides a model category whose fibrant objects are m-hybrid n-fold Segal spaces.

One can show (see e.g. in [Bar05, Lur09b, BR13, BR20]) that equivalences between (possibly non-complete) n-fold Segal spaces for this model structure are exactly the *Dwyer-Kan equivalences*, which are defined inductively. For this we need the following inductive definition of the homotopy category of an n-fold Segal space:

Definition 1.3.11. The *homotopy category* $h_1(X)$ of an n-fold Segal space $X_{\bullet,\dots,\bullet}$ is the following category: its objects are the 0-simplices, i.e. the points of $X_{0,\dots,0}$. For x,y two objects, we let

$$\mathrm{Hom}_X(x,y)_{\bullet,\dots,\bullet} := \{x\} \overset{h}{\underset{X_{0,\bullet,\dots,\bullet}}{\times}} X_{1,\bullet,\dots,\bullet} \overset{h}{\underset{X_{0,\bullet,\dots,\bullet}}{\times}} \{y\}$$

be the $(n-1)$-fold Segal space of morphisms[k] from x to y. Now let morphisms in $h_1(X)$ from x to y be the set of isomorphism classes of objects in $h_1(\mathrm{Hom}_X(x,y)_{\bullet,\dots,\bullet})$, which is already defined by induction. Composition is defined using the Segal condition in the first index.

Definition 1.3.12. A morphism $f : X \to Y$ of n-fold Segal spaces is a *Dwyer-Kan equivalence* if

1. the induced functor $h_1(f) : h_1(X) \to h_1(Y)$ is essentially surjective.

[k]We will revisit this notion in 1.3.4.

2. for each pair of objects $x, y \in X_{0,\dots,0}$, the induced morphism $\mathrm{Hom}_X(x,y) \to \mathrm{Hom}_Y(f(x), f(y))$ is a Dwyer-Kan equivalence of $(n-1)$-fold Segal spaces.

Again we obtain equivalences of complete Segal spaces

$$N(\mathcal{CSSp}_n, \ell we) \longrightarrow N(\mathcal{S}e\mathcal{S}p_n, \mathcal{DK}) \longrightarrow N(s\mathcal{S}pace_n, \mathcal{W}_f^{CSe}),$$

where $\mathcal{W}_f^{CSe}$ is the subcategory of weak equivalences in the localization $s\mathcal{S}pace_{n,f}^{CSe}$.

Remark 1.3.13. Note that $\mathcal{CSSp}_n$ is the subcategory of fibrant objects for a left Bousfield localization of $s\mathcal{S}pace_{n,f}$ and weak equivalences of complete n-fold Segal spaces are level-wise weak equivalences. Denoting the category of fibrant objects in $s\mathcal{S}pace_{n,c}^{CSe}$, the Reedy fibrant complete n-fold Segal spaces, by $\mathcal{CSSp}_{n,c}$, the Quillen equivalence between $s\mathcal{S}pace_{n,c}$ and $s\mathcal{S}pace_{n,f}$ induces an equivalence $N(\mathcal{CSSp}_{n,c}, \ell we) \longrightarrow N(\mathcal{CSSp}_n, \ell we)$, whose inverse is given by Reedy fibrant replacement $(-)^R$.

Recall from Remark 1.3.5 that we can think of an n-fold Segal space in an iterative way: we can view an n-fold Segal space as a Segal object in $(n-1)$-fold Segal spaces, which we in turn can think of a Segal object in Segal objects in $(n-2)$-fold Segal spaces, etc. Then condition (CSS^i) above means that the ith iteration is a *complete* Segal space object. For more on this point of view, see [Lur09b, Hau18]

Definition 1.3.14. Given an n-fold Segal space $X_{\bullet,\dots,\bullet}$, one can apply the completion functor iteratively to obtain a complete n-fold Segal space $\widehat{X}_{\bullet,\dots,\bullet}$, its *(n-fold) completion*. This yields a map $X \to \widehat{X}$, the *completion map*, which is universal among all maps (in the homotopy category) to complete n-fold Segal spaces. It is a left adjoint to the embedding of $\mathcal{CSSp}_n[\ell we^{-1}]$ into $\mathcal{S}e\mathcal{S}p_n[\ell we^{-1}]$.

If an n-fold Segal space $X_{\bullet,\dots,\bullet}$ satisfies (SC^j) for $j \leqslant m$, we can apply the completion functor just to the last $(n-m)$ indices to obtain an m-hybrid n-fold Segal space $\widehat{X}^m_{\bullet,\dots,\bullet}$, its *m-hybrid completion*.

1.3.4 Constructions of n-fold Segal spaces

We describe several intuitive constructions of (∞,n)-categories in terms of (complete) n-fold Segal spaces.

Truncation

*Given an (∞,n)-category, for $k \leqslant n$ its (∞,k)-*truncation, or k-truncation, *is the (∞,k)-category obtained by discarding the non-invertible m-morphisms for $k < m \leqslant n$.*

In terms of n-fold Segal spaces, there is a functor $\tau_k : \mathscr{S}e\mathscr{S}p_n \to \mathscr{S}e\mathscr{S}p_k$ sending $X = X_{\bullet,\dots,\bullet}$ to its k-truncation, the k-fold Segal space

$$\tau_k X = X_{\underbrace{\bullet,\dots,\bullet}_{k \text{ times}},\underbrace{0,\dots,0}_{n-k \text{ times}}}.$$

If X is m-hybrid then so is $\tau_k X$ by the definition of the conditions (1.3.7) and (1.3.7). In particular, if X is complete, then $\tau_k X$ is as well, and thus, the truncation of an (∞,n)-category is an (∞,k)-category.

Caution 1.3.15. Truncation does not behave well with respect to completion, i.e. the truncation of the completion is not the completion of the truncation. However, we get a map in one direction:

$$\begin{tikzcd} \tau_k(X) \arrow[r] \arrow[d] & \tau_k(\widehat{X}) \\ \widehat{\tau_k(X)} \arrow[ur, dashed] & \end{tikzcd}$$

In general, this map is not an equivalence. So in general one should always complete an n-fold Segal space before truncating it. For example, for $n = 1$ and a non-complete Segal Space X, the truncation $\tau_1(X) = X_0$ is just the zeroth space, but the truncation of the completion will be equivalent to the underlying ∞-groupoid X_1^{inv}. The map in this case is given by the degeneracy map. In the example $X = N(G)$ from Remark 1.2.19, the former is $N(G)_0 = \{*\}$ and the latter is BG, which are not equivalent in general.

Remark 1.3.16. As explained above, the k-truncation of an (∞,n)-category X should be the maximal (∞,k)-category contained in X. However, the image of the degeneracy

$$X_{\underbrace{1,\dots,1}_{k},0,\dots,0} \hookrightarrow X_{\underbrace{1,\dots,1}_{m},0,\dots,0}$$

consists exactly of the invertible m-morphisms for $k < m \leqslant n$ if and only if X satisfies (1.3.7) for $k < i \leqslant n$. For example, if $X = X_\bullet$ is a (1-fold) Segal space then X_0 is the underlying ∞-groupoid of invertible morphisms if and only if X is complete.

Extension

Any (∞,n)-category can be viewed as an $(\infty,n+1)$-category with only identities as $(n+1)$-morphisms.

In terms of n-fold Segal spaces, any n-fold Segal space can be viewed as a constant simplicial object in n-fold Segal spaces, i.e. an $(n+1)$-fold Segal space which is constant in the first index. Explicitly, if $X_{\bullet,\dots,\bullet}$ is an n-fold Segal space, then $\varepsilon(X)_{\bullet,\dots,\bullet}$ is the constant simplicial object in the category of Segal spaces given by X, i.e. it is the $(n+1)$-fold Segal space such that for every $k \geqslant 0$,

$$\varepsilon(X)_{\bullet,\dots,\bullet,k} = X_{\bullet,\dots,\bullet}$$

and the face and degeneracy maps in the last index are identity maps.

Lemma 1.3.17. *If X is complete, then $\varepsilon(X)$ is complete.*

Proof. Since X is complete, it satisfies (CSS^i) for $i > 1$. For $i = 0$, we have to show that $\varepsilon(X)_{\bullet,0,\dots,0}$ is complete. This is satisfied because

$$(\varepsilon(X)_{1,0,\dots,0})^{inv} = \varepsilon(X)_{1,0,\dots,0} = X_{0,\dots,0} = \varepsilon(X)_{0,0,\dots,0},$$

since morphisms between two elements x, y in the homotopy category of $\varepsilon(X)_{\bullet,k_2,\dots,k_n}$ are just connected components of the space of paths in $X_{k_2,\dots,k_n}$, and thus are always invertible. □

We call ε the extension functor, which is left adjoint to τ_n. Moreover, the unit $\mathrm{id} \to \tau_1 \circ \varepsilon$ of the adjunction is the identity.

Inverting

Given an (∞, n)-category, for $k \leqslant n$ we obtain an (∞, k)-category by inverting the non-invertible m-morphisms for $k < m \leqslant n$.

We saw that the extension functor ε had a right adjoint τ_n. It also has a left adjoint η, which formally inverts all $(n+1)$-morphisms. For an n-fold Segal space X, this is given by realizing the last index,

$$(\eta X)_{k_1,\dots,k_n} = |X_{k_1,\dots,k_n,\bullet}|.$$

Here geometric realization amounts to taking the diagonal of the bisimplicial set $X_{k_1,\dots,k_n,\bullet}$. Since the following diagram of right adjoints commutes, the diagram of left adjoints commutes as well. Therefore, completion and inverting commute.

$$\begin{tikzcd}
\mathscr{S}e\mathscr{S}p_{n+1} \arrow[r, bend right, "\eta"'] \arrow[r, phantom, "\top"] \arrow[d, bend right, "\widehat{(-)}"'] \arrow[d, phantom, "\dashv"] & \mathscr{S}e\mathscr{S}p_n \arrow[l, bend right, "\varepsilon"'] \arrow[d, bend right, "\widehat{(-)}"'] \arrow[d, phantom, "\dashv"] \\
\mathscr{C}\mathscr{S}\mathscr{S}p_{n+1} \arrow[u, bend right] \arrow[r, bend right, "\eta"'] \arrow[r, phantom, "\top"] & \mathscr{C}\mathscr{S}\mathscr{S}p_n \arrow[u, bend right] \arrow[l, bend right, "\varepsilon"']
\end{tikzcd}$$

The higher category of morphisms and loopings

Given two objects x, y in an (∞, n)-category, morphisms from x to y should form an $(\infty, n-1)$-category.

This can be realized for n-fold Segal spaces, which is one of the main advantages of this model for (∞, n)-categories.

Definition 1.3.18. Let $X = X_{\bullet,\cdots,\bullet}$ be an n-fold Segal space. As we have seen above one should think of objects as vertices of the space $X_{0,\dots,0}$. Let $x, y \in X_{0,\dots,0}$. The *$(n-1)$-fold Segal space of morphisms from x to y* is

$$\mathrm{Hom}_X(x,y)_{\bullet,\cdots,\bullet} = \{x\} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} X_{1,\bullet,\cdots,\bullet} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} \{y\}\,.$$

Remark 1.3.19. Note that if X is m-hybrid, then $\mathrm{Hom}_X(x,y)$ is $(m-1)$-hybrid.

Example 1.3.20 (Compatibility with extension)**.** Let X be an $(\infty,0)$-category, i.e. a space, viewed as an $(\infty,1)$-category, i.e. a constant (complete) Segal space $\varepsilon(X)_\bullet$, $\varepsilon(X)_k = X$. For any two objects $x, y \in \varepsilon(X)_0 = X$ the $(\infty,0)$-category, i.e. the space, of morphisms from x to y is

$$\mathrm{Hom}_{\varepsilon(X)}(x,y) = \{x\} \underset{\varepsilon(X)_0}{\overset{h}{\times}} \varepsilon(X)_1 \underset{\varepsilon(X)_0}{\overset{h}{\times}} \{y\} = \{x\} \underset{X}{\overset{h}{\times}} \{y\} = \mathrm{Path}_X(x,y)\,,$$

the path space in X, which coincides with what one expects by the interpretation of paths, homotopies, homotopies between homotopies, etc. being higher invertible morphisms.

Definition 1.3.21. Let X be an n-fold Segal space, and $x \in X_0$ an object in X. Then the *looping of X at x* is the $(n-1)$-fold Segal space

$$\Omega_x(X)_{\bullet,\dots,\bullet} = \mathrm{Hom}_X(x,x)_{\bullet,\dots,\bullet} = \{x\} \times^h_{X_{0,\bullet,\dots,\bullet}} X_{1,\bullet,\dots,\bullet} \times^h_{X_{0,\bullet,\dots,\bullet}} \{x\}.$$

In the following, it will often be clear at which element we are looping, e.g. if there essentially is only one element, or at a unit for a monoidal structure, which we define in the next section. Then we omit the x from the notation and just write

$$\Omega X = \Omega(X) = \Omega_x(X).$$

We can iterate this procedure as follows.

Definition 1.3.22. Let $\Omega^0_x(X) = X$. For $1 \leqslant k \leqslant n$, let the *k-fold iterated looping* be the $(n-k)$-fold Segal space

$$\Omega^k_x(X) = \Omega_x(\Omega^{k-1}_x(X)),$$

where we view x as a trivial k-morphism via the degeneracy maps, i.e. an element in $\Omega^{k-1}_x(X)_{0\dots,0} \to X_{\underbrace{1,\dots,1}_{k},0,\dots,0}$.

Looping k times commutes with taking the k-hybrid completion up to weak equivalence, since completion is taken index by index:

Let X be a k-hybrid n-fold Segal space. Then for the k-hybrid completion $\hat{X}$, which is the completion in the last $(n-k)$ variables, we have that $\Omega^k(\hat{X}) \xrightarrow{\simeq} \hat{X}_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet}$ is complete, so by the universal property of completion, the horizontal map in the following diagram exists:

$$\begin{tikzcd}
\widehat{\Omega^k(X)} \arrow[r, dashed] & \Omega^k(\hat{X}) \\
\Omega^k(X) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Lemma 1.3.23. *Let X be a k-hybrid n-fold Segal space. Then the induced map*

$$\widehat{\Omega^k(X)} \xrightarrow{\simeq} \Omega^k(\hat{X})$$

is a level-wise weak equivalence.

Proof. In the diagram

$$\begin{tikzcd}
\widehat{\Omega^k(X)} \arrow[r, dashed] & \Omega^k(\hat{X}) \\
\Omega^k(X) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

we know that the vertical map is a DK-equivalence, since completions are DK-equivalences. Moreover, since X is hybrid, we have that $\Omega^k(X) \xrightarrow{\simeq} X_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet}$ and $\Omega^k(\hat{X}) \xrightarrow{\simeq} \hat{X}_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet}$, and by definition of (hybrid) completion, $X_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet} \to \hat{X}_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet}$ is just a completion, so it is a DK-equivalence. Thus, in the diagram

$$\begin{tikzcd}
X_{\widehat{\underbrace{1,\dots 1}_{k},\bullet},\dots,\bullet} \arrow[r, dashed] & \hat{X}_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet} \\
X_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

by the two-out-of-three property, the horizontal morphism is as well. But since both $X_{\widehat{\underbrace{1,\dots 1}_{k},\bullet},\dots,\bullet}$ and $\hat{X}_{\underbrace{1,\dots 1}_{k},\bullet,\dots,\bullet}$ are complete, it is a level-wise equivalence. $\square$

n-fold from n-uple Segal spaces

We can extract the maximal n-fold Segal space from an n-uple one by the following procedure. Let us recall and introduce some notation for various model structures on the category of n-uple simplicial spaces.

- $s\mathscr{S}pace^{(C)Se}_{n,f}$, where fibrant objects are (complete) n-fold Segal spaces.
- $s\mathscr{S}pace^{(C)Se}_{n,c}$, where fibrant objects are Reedy fibrant (complete) n-fold Segal spaces.
- $s\mathscr{S}pace^{n,f}_{Se}$, where fibrant objects are n-uple Segal spaces.
- $s\mathscr{S}pace^{n,c}_{Se}$, where fibrant objects are Reedy fibrant n-uple Segal spaces.

From now, let $* \in \{c, f\}$. There are (two) Quillen adjunctions

$$s\mathscr{S}pace^{Se}_{n,*} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\leftrightarrows}} s\mathscr{S}pace^{n,*}_{Se}\ .$$

Let us denote (in a rather unusual way) $\mathbf{L} := \mathbb{R}\mathrm{id} : N(s\mathscr{S}pace^{Se}_{n,*}, w.e.) \to N(s\mathscr{S}pace^{n,*}_{Se}, w.e.)$. Observe that on fibrant objects, $\mathbf{L}$ is nothing but the inclusion of (possibly Reedy fibrant) n-fold Segal spaces into (possibly Reedy fibrant) n-uple Segal spaces. After [Hau18, Proposition 4.12], we know it has a right adjoint $\mathbf{R}$. For the given (possibly Reedy fibrant) n-uple Segal space X, we wish to compute $\mathbf{R}(X)$. By adjunction, we know that

$$\mathbf{R}(X)_{\bullet,\dots,\bullet} \simeq \mathrm{Map}^h_{s\mathscr{S}pace^{Se}_{n,*}}\left(\Delta^{\vec{\bullet}}, \mathbf{R}(X)\right) \simeq \mathrm{Map}^h_{\mathbf{s\mathscr{S}pace}^{n,*}_{Se}}\left(\mathbf{L}(\Delta^{\vec{\bullet}}), X\right),$$

where $\Delta^{\vec{k}}$ for $\vec{k} = (k_1, \dots, k_n)$ is the n-fold simplicial set represented by $[k_1] \times \cdots \times [k_n] \in \Delta^{\times n}$, and Map^h denotes the derived mapping space.

We will now find an explicit way to compute $\mathbf{R}(X)$ by finding cofibrant replacements of $\mathbf{L}(\Delta^{\vec{k}})$. We start by recalling certain strict n-categories of the desired shapes, which are all objects in Joyal's category Θ_n [Rez10].

For $\vec{k} = (k_1, \dots, k_n)$, let $\Theta^{\vec{\bullet}}$ be the *walking $\vec{k}$-tuple of n-morphisms* which is the strict n-category from [JFS17, Definition 5.1]. We do not want to recall the full definition here, but rather the intuition:

- For $\vec{k} = (1, 0, \dots, 0)$, the category $\Theta^{\vec{k}} = \bullet \to \bullet$ is the walking 1-morphism.
- For $\vec{k} = (2, 0, \dots, 0)$, the category $\Theta^{\vec{k}} = \bullet \to \bullet \to \bullet$ is the walking composable pair of 1-morphisms.
- For $\vec{k} = (2, 1, \dots, 0)$, the strict 2-category $\Theta^{\vec{k}} = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & \bullet \end{tikzcd}$ is the walking horizontally composable pair of 2-morphisms.

- For $\vec{k} = (3,2,\ldots,0)$, we have the strict 2-category $\Theta^{\vec{k}} = \begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Uparrow"] \arrow[r, phantom, shift right=2.5ex, "\Uparrow"] & \bullet \end{tikzcd}$

- More generally, for $\vec{k} = (k_1,\ldots,k_n)$, the strict n-category $\Theta^{\vec{\bullet}}$ has $k_1 \cdots k_n$ n-morphisms which are composable following the pattern of a grid of dimension $k_1 \times \cdots \times k_n$.

The elementary building blocks for these categories are $\Theta^{(n)}$, where $(n) = (\underbrace{1,\ldots,1}_{n},0,\ldots,0)$. All others are built by gluing these in a grid of of dimension $k_1 \times \cdots \times k_n$. In [BSP21] Barwick–Schommer-Pries use the following definition, which can been easily seen to be equivalent to the one in [JFS17] by induction:

Definition 1.3.24. Let C^1 be the walking 1-morphism, i.e. the category with two objects and one non-identity morphism from one object to the other, $C^1 = \{\bullet \longrightarrow \bullet\}$. The strict n-category $\Theta^{(n)}$ is defined inductively by the pushout square

$$\begin{tikzcd} \{0,1\} \times \Theta^{(n-1)} \arrow[r, hook] \arrow[d] & C^1 \times \Theta^{(n-1)} \arrow[d] \\ \{0,1\} \times \{*\} \arrow[r] & \Theta^{(n)}. \end{tikzcd}$$

Note that this immediately implies the existence of a surjective "collapse" map $c_n : C^n \to \Theta^{(n)}$, where $C^n = (C^1)^{\times n}$ is the walking n-morphism as a strict n-uple category.

The n-fold nerve of $\Theta^{\vec{k}}$ is

- levelwise fibrant (because $\Theta^{\vec{k}}$ is discrete).
- a Segal space (because $\Theta^{\vec{k}}$ is a strict n-category).
- complete (because $\Theta^{\vec{k}}$ is reduced).

Let us thus abuse notation and still write $\Theta^{\vec{k}}$ for this (complete) n-fold Segal space. Now we can write the formula for the cofibrant replacement, and therefore the recipe for finding the underlying n-fold Segal space.

Theorem 1.3.25. *Given a n-uple Segal space X, its maximal underlying n-fold Segal space has levels, for $\vec{k} = (k_1,\ldots,k_n) \in (\Delta^{op})^n$,*

$$\mathcal{R}(X)_{\vec{k}} = \mathrm{Map}^h_{s\mathcal{S}pace^{n,*}_{Se}}(\Theta^{\vec{k}}, X).$$

Since $\Theta^{\vec{\bullet}}$ is an n-fold cosimplicial object in strict n-categories (see [JFS17]), this defines a (complete) n-fold Segal space.

To prove this Theorem, we need to understand what the cofibrant replacement $\mathbf{L}(\Delta^{\vec{\bullet}})$ is. The first step is a tool to compute the right hand expression in the Theorem, namely, an explicit cofibrant replacement of $\Theta^{\vec{k}}$.

Proposition 1.3.26. *For $n = 1$, the category $\Theta^{(1)}$, or rather its nerve, is cofibrant in the projective model structure $s\mathscr{S}pace^{1,f}_{CSe}$. For $n > 1$, a cofibrant replacement of $\Theta^{(n)}$ in the projective model structure of n-uple Segal spaces $s\mathscr{S}pace^{n,f}_{Se}$ is given inductively by replacing the pushouts in the definition by homotopy pushouts and $\Theta^{(n-1)}$ by its (inductively already defined) cofibrant replacement.*

Proof. Similarly to Section 1.2.4, we use an argument similar to that in [JFS17], Remark 3.4., which observes the following: $\Theta^{(2)}$ is given by a strict pushout along a diagram of cofibrant objects of which one arrow is an inclusion. By [Lur09a, A.2.4.4], this is a homotopy pushout in the injective model structure and therefore homotopy equivalent to the homotopy pushout in the projective model structure. So a cofibrant replacement of $\Theta^{(2)}$ is given by taking the homotopy pushout of the same diagram,

$$\begin{tikzcd} \{0,1\} \times C^1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\mathrm{h}\lrcorner"] & C^2 \arrow[d] \\ \{0,1\} \times \{*\} \arrow[r] & \mathrm{cof}(\Theta^{(2)}) \end{tikzcd}$$

Now we proceed by induction. Assume we have shown the statement for $k < n$ and we have a cofibrant replacement $\mathrm{cof}(\Theta^{(k)})$ given as in the Proposition. Then, since the map $\{0,1\} \hookrightarrow C^1$ is a cofibration in the projective model structure, the map $\{0,1\} \times \mathrm{cof}(\Theta^{(n-1)}) \hookrightarrow C^1 \times \mathrm{cof}(\Theta^{(n-1)})$ is a cofibration. Moreover, $\{0,1\} \times \mathrm{cof}(\Theta^{(n-1)})$, $C^1 \times \mathrm{cof}(\Theta^{(n-1)})$, and $\{0,1\} \times \{*\}$ are all cofibrant, so we can use the above-mentioned [Lur09a, A.2.4.4], again to see that the strict pushout, which is weakly equivalent to $\Theta^{(n)}$, is a homotopy pushout, and moreover cofibrant. Summarizing, it is a cofibrant replacement of $\Theta^{(n)}$. □

Remark 1.3.27. Similarly, we can obtain cofibrant replacements for $\Theta^{\vec{k}}$ as defined in [JFS17] by replacing the pushouts in the definition by homotopy pushouts.

The remaining ingredient in the proof of the Theorem is the following Lemma.

Lemma 1.3.28. *The natural map $\Delta^{\vec{k}} \to \Theta^{\vec{k}}$ is a weak equivalence in $s\mathscr{S}pace^{Se}_{n,*}$.*

Proof. We need to show that for any fibrant object Y in $s\mathscr{S}pace^{Se}_{n,*}$ the induced map $\mathrm{Map}^h_{s\mathscr{S}pace^{Se}_{n,*}}(\Theta^{\vec{k}}, Y) \to \mathrm{Map}^h_{s\mathscr{S}pace^{Se}_{n,*}}(\Delta^{\vec{k}}, Y)$ is a weak equivalence of simplicial sets.

We show the claim for $\vec{k}=(k)$ proceeding by induction using the explicit cofibrant replacement from the previous Proposition. For $k=1$, this is true, since $\Theta^{(1)}=\Delta^{(1)}=\Delta^1$. Assume we have proven the statement for $l<k$. Then

$$\begin{aligned}
\mathrm{Map}^h(\Theta^{(k)},Y) \xrightarrow{\simeq} & \mathrm{Map}^h(C^1\times\Theta^{(k-1)},Y) \overset{h}{\underset{\mathrm{Map}^h(\{0,1\}\times\Theta^{(k-1)},Y)}{\times}} \mathrm{Map}^h(\{0,1\},Y)\\
&\simeq \mathrm{Map}^h(C^1\times\Theta^{(k-1)},Y) \overset{h}{\underset{Y^{\times 2}_{0,\bullet,\dots,\bullet}}{\times}} Y^{\times 2}_{0,\dots,0}\\
&\simeq \mathrm{Map}^h(C^1\times\Theta^{(k-1)},Y)\\
&\simeq \mathrm{Map}^h(\Theta^{(k-1)},\mathrm{Hom}(C^1,Y)).
\end{aligned}$$

Here the first equivalence uses that the cofibrant replacement of $\Theta^{(k)}$ is the homotopy pushout as described in the previous Proposition, the next equivalence computes the mapping spaces on the right and below the times symbol, the third equivalence uses essential constancy of Y, i.e. condition (ii) in Definition 1.3.4, and the last one uses that n-fold Segal spaces are Cartesian closed.

By the induction hypothesis, the natural map $\Delta^{(k-1)}\to\Theta^{(k-1)}$ induces an equivalence

$$\mathrm{Map}^h(\Theta^{(k-1)},\mathrm{Hom}(C^1,Y)) \xrightarrow{\simeq} \mathrm{Map}^h(\Delta^{(k-1)},\mathrm{Hom}(C^1,Y)) \simeq \mathrm{Map}^h(\Delta^{(k)},Y)\simeq Y_{(k)}.$$

A similar argument works for general $\vec{k}$. □

Remark 1.3.29. The above Lemma is equivalent to the observation that the model structure $s\mathscr{S}pace^{Se}_{n,*}$ can be obtained as the left Bousfield localization of $s\mathscr{S}pace^{n,*}_{Se}$ along $\Delta^{\vec{k}}\to\Theta^{\vec{k}}$.

Proof of Theorem 1.3.25. The following equivalences are compatible with the cosimplicial structure of $\Delta^{\vec{\bullet}}$ and $\Theta^{\vec{\bullet}}$:

$$\mathbf{R}(X)_{\vec{k}} \cong \mathrm{Map}^h\big(\Delta^{\vec{k}},\mathbf{R}(X)\big) \simeq \mathrm{Map}^h\big(\mathbf{L}(\Delta^{\vec{k}}),X\big) \xleftarrow[\simeq]{\text{Lemma 1.3.28}} \mathrm{Map}^h\big(\mathbf{L}(\Theta^{\vec{k}}),X\big) \simeq \mathrm{Map}^h(\Theta^{\vec{k}},X).$$

□

Chapter 2

Higher bordism Categories

Section 2.1

The complete n-fold Segal space of closed intervals

In this section we define a complete Segal space $\mathrm{Int}_\bullet$ of closed intervals in $\mathbb{R}$ which will form the basis of the n-fold Segal space of bordisms. It will be a tool to record where (in the time direction) the bordisms can be cut. In particular, there will be a forgetful functor from bordisms to these closed intervals. We start by defining an internal category of closed intervals in $\mathbb{R}$, whose nerve will give a complete Segal space of certain tuples of closed intervals. However, for our model of the bordism category, to avoid having to deal with manifolds with corners, we will instead want to interpret the tuples of intervals as being closed in an open interval of finite length (instead of $\mathbb{R}$). This will be explained in 2.1.3. Finally, we could have chosen that open interval to always be $(0,1)$ and thus fix the "length" in the time direction of the bordism and its collars to be 1. This choice requires rescaling and will be explained in 2.1.5.

2.1.1 Int^c as an internal category

We first define a category internal to topological spaces $\mathfrak{Int}_c$ which gives rise to a strongly Segal internal category Int^c of closed intervals in $\mathbb{R}$.

The topological space of objects of $\mathfrak{Int}_c$ is

$$\mathfrak{Int}_0^c = \{(a,b) : a < b\} \subset \mathbb{R}^2 \tag{2.1.1}$$

with the standard topology from $\mathbb{R}^2$. We interpret an element $(a,b) \in \mathfrak{Int}_0^c$ as the closed interval $I = [a,b]$. This interpretation gives a bijection from the set of points of the topological space $\mathfrak{Int}_0^c$ to the set of closed bounded intervals:

$$\mathfrak{Int}_0^c \longleftrightarrow \{\text{closed bounded intervals } I = [a,b] \text{ in } \mathbb{R} \text{ with non-empty interior}\}$$

which we use as an identification. In fact, $\mathfrak{Int}_0^c$ is a submanifold of $\mathbb{R}^2$ and to get the desired Kan complex Int_0^c, we take smooth singular simplices (see e.g. [Lee13]), i.e. for $l \geqslant 0$, the l-simplices are pairs of smooth maps $a,b : |\Delta^l|_e \to \mathbb{R}$ such that $a(s) < b(s)$ for every $s \in |\Delta^l|_e$. Faces and degeneracies are the usual ones. We view such an l-simplex as a *closed interval bundle* and denote it by $[a,b] \to |\Delta^l|_e$ or $(I(s))_{s\in|\Delta^l|_e} = (a(s),b(s))_{s\in|\Delta^l|_e}$.

The topological space of morphisms of $\mathfrak{Int}_c$ is

$$\mathfrak{Int}_1^c = \{(a_0,a_1,b_0,b_1) : a_j < b_j \text{ for } j = 0,1, \text{ and } a_0 \leqslant a_1, b_0 \leqslant b_1\} \subset \mathbb{R}^4, \tag{2.1.2}$$

again with the standard topology from $\mathbb{R}^4$. Now we interpret an element $(a_0,a_1,b_0,b_1) \in \mathfrak{Int}_1^c$ as a pair of ordered closed intervals $I_0 \leqslant I_1$, where $I_0 = [a_0,b_0]$ and $I_1 = [a_1,b_1]$. Here "ordered" means that $a_0 \leqslant a_1$ and $b_0 \leqslant b_1$. This gives an identification of the points of the topological space with certain pairs of intervals:

$$\mathfrak{Int}_1^c \longleftrightarrow \{I_0 \leqslant I_1 : I_j = [a_j, b_j] \text{ with } a_j < b_j \text{ for } j = 0,1, \text{ and } a_0 \leqslant a_1, b_0 \leqslant b_1\}.$$

As above $\mathfrak{Int}_1^c$ has the structure of a submanifold of $\mathbb{R}^4$ and by taking smooth singular simplices we obtain a Kan complex Int_1^c whose l-simplices now are quadruples of smooth maps $a_0, a_1, b_0, b_1 : |\Delta^l|_e \to \mathbb{R}$ such that $a_j(s) < b_j(s)$ for $j = 0,1$, $a_0(s) \leqslant a_1(s)$, and $b_0(s) \leqslant b_1(s)$ for every $s \in |\Delta^l|_e$. We view such an l-simplex as a *closed interval bundle with two closed subintervals* and denote it by $([a_0, b_0] \leqslant [a_1, b_1]) \to |\Delta^l|_e$ or $(I_0(s) \leqslant I_1(s))_{|\Delta^l|_e}$.

The face and degeneracy maps

$$\begin{tikzcd} \mathfrak{Int}_0^c \arrow[r, "d" description] & \mathfrak{Int}_1^c \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd}$$

arise from forgetting and repeating an interval, respectively:

$$s : [a_0, b_0] \leqslant [a_1, b_1] \longmapsto [a_0, b_0],$$

$$t : [a_0, b_0] \leqslant [a_1, b_1] \longmapsto [a_1, b_1],$$

and

$$d : [a, b] \longmapsto [a, b] \leqslant [a, b].$$

Composition is given by remembering the outer intervals:

$$([a_0, b_0] \leqslant [a_1, b_1]) \circ ([a_1, b_1] \leqslant [a_2, b_2]) = ([a_0, b_0] \leqslant [a_2, b_2]).$$

Here s, t, and d are smooth maps, so $\mathfrak{Int}^c$ is a category internal to manifolds. Thus, when taking smooth singular simplices to get Int^c, all above assignments are well-defined for l-simplices as well and commute with the faces and degeneracies. Moreover, s and t are fibrations since they are restrictions of projections.

Remark 2.1.1. Note that even though we like to think of the l-simplices in Int_0^c and Int_1^c as "closed interval bundles", we do not treat them as such: face and degeneracy maps are not defined to be pullbacks of the bundles, which would only be defined up to isomorphism; instead, they are defined explicitly at the level of spaces to ensure that simplicial functoriality holds.

Summarizing, we obtain

Lemma 2.1.2. Int^c *is a strongly Segal internal category* [a].

Moreover, the spaces of objects and morphisms are contractible:

[a]A *strongly Segal internal category* is a category $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ internal to $\mathscr{S} = \mathscr{S}pace \subset s\mathscr{S}et$ such that the source and target maps $s, t : \mathscr{C}_1 \to \mathscr{C}_0$ are fibrations of simplicial sets.

Lemma 2.1.3. $\mathrm{Int}_0^c \simeq \mathrm{Int}_1^c \simeq *$.

Proof. The underlying topological space is contractible as a subspace of $\mathbb{R}^{2k}$, so the associated Kan complex given by taking smooth simplices is also contractible. ☐

2.1.2 Int^c as a complete Segal space

We defined Int^c as a strongly Segal internal category in the previous section. Its nerve is a Segal space $\mathrm{Int}_\bullet^c = N(\mathrm{Int}^c)_\bullet$ (abuse of notation). Let us spell out this Segal space in more detail to become more familiar with it.

For an integer $k \geqslant 0$, let

$$\mathfrak{Int}_k^c = \{(\underline{a},\underline{b}) = (a_0,\dots,a_k,b_0,\dots,b_k) : a_j < b_j \text{ for } 0 \leqslant j \leqslant k, \text{ and } \\ a_{j-1} \leqslant a_j \text{ and } b_{j-1} \leqslant b_j \text{ for } 1 \leqslant j \leqslant k\} \subset \mathbb{R}^{2k} \tag{2.1.3}$$

with the subspace topology. As above, one can extract Kan complexes Int_k^c by taking smooth simplices. Note that for $k = 0,1$ this coincides with (2.1.1) and (2.1.2) above. As before, we interpret an element $(\underline{a},\underline{b})$ as an ordered $(k+1)$-tuple of closed intervals $\underline{I} = I_0 \leqslant \cdots \leqslant I_k$ with left endpoints a_j and right endpoints b_j such that I_j has non-empty interior. By "ordered", i.e. $I_j \leqslant I_{j'}$, we mean that the endpoints are ordered, i.e. $a_j \leqslant a_{j'}$ and $b_j \leqslant b_{j'}$ for $j \leqslant j'$.

Spatial structure of the levels The spatial structure of a level Int_k^c comes from taking smooth singular simplices of the submanifold of $\mathbb{R}^{2k}$. Thus, an l-simplex consists of smooth maps

$$|\Delta^l|_e \to \mathbb{R}, \quad s \mapsto a_j(s), b_j(s)$$

for $j = 0,\dots,k$ such that for every $s \in |\Delta^l|_e$, the following inequalities hold:

$$\begin{aligned} a_i(s) < b_i(s), &\quad \text{for } i = 0,\dots,k \\ a_{i-1}(s) \leqslant a_i(s), &\quad \text{and} \\ b_{i-1}(s) \leqslant b_i(s) &\quad \text{for } i = 1,\dots,k. \end{aligned}$$

We denote an l-simplex by $(I_0 \leqslant \cdots \leqslant I_k) \to |\Delta^l|_e$ or $(I_0(s) \leqslant \cdots \leqslant I_k(s))_{s\in|\Delta^l|}$ and call it a *closed interval bundle with* $(k+1)$ *subintervals*.

For a morphism $f : [m] \to [l]$ in the simplex category Δ, i.e. a (weakly) order-preserving map, let $|f| : |\Delta^m|_e \to |\Delta^l|_e$ be the induced map between standard simplices. Let f^Δ be the map sending an l-simplex in Int_k^c to the m-simplex in Int_k^c given by precomposing with $|f|$,

$$f^\Delta : \big(I_0(s) \leqslant \cdots \leqslant I_k(s)\big)_{s\in|\Delta^l|_e} \longmapsto \big(I_0(|f|(s)) \leqslant \ldots \leqslant I_k(|f|(s))\big)_{s\in|\Delta^m|_e}.$$

Notation 2.1.4. We denote the *spatial face and degeneracy maps* of Int^c_k by d^Δ_j and s^Δ_j for $0 \leqslant j \leqslant l$.

The following Lemma is a straightforward generalization of Lemma 2.1.3.

Lemma 2.1.5. *Each level* Int^c_k *is a contractible Kan complex.*

Simplicial structure – the simplicial space $\mathrm{Int}^c_\bullet$ By construction, since Int^c was strongly Segal, its nerve is a functor $\mathrm{Int}^c_\bullet : \Delta^{op} \to \mathscr{S}pace$. Let us recall that to a morphism $g : [m] \to [k]$ in Δ, it assigns

$$\begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{g^*} & \mathrm{Int}_m, \\ (I_0(s) \leqslant \cdots \leqslant I_k(s))_{s \in |\Delta^l|_e} & \longmapsto & (I_{g(0)}(s) \leqslant \cdots \leqslant I_{g(m)}(s)))_{s \in |\Delta^l|_e}. \end{array}$$

One could alternatively see this directly by observing that the assignment is clearly functorial and f^Δ and g^* commute for all morphisms f, g in Δ.

Notation 2.1.6. We denote the *simplicial face and degeneracy maps* by d_j and s_j for $0 \leqslant j \leqslant k$.

Explicitly, they are given by the following formulas. The jth degeneracy map is given by doubling the jth interval, and the jth face map is given by deleting the jth interval,

$$\begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{s_j} & \mathrm{Int}_{k+1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & I_0 \leqslant \cdots \leqslant I_j \leqslant I_j \leqslant \cdots \leqslant I_k, \end{array} \qquad \begin{array}{rcl} \mathrm{Int}_k & \xrightarrow{d_j} & \mathrm{Int}_{k-1}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & I_0 \leqslant \cdots \leqslant \hat{I}_j \leqslant \cdots \leqslant I_k. \end{array}$$

The complete Segal space $\mathrm{Int}^c_\bullet$

Proposition 2.1.7. $\mathrm{Int}^c_\bullet$ *is a complete Segal space. Moreover, the inclusion* $* \hookrightarrow \mathrm{Int}^c_\bullet$ *given by degeneracies, where* $*$ *is seen as a constant complete Segal space, is an equivalence of complete Segal spaces.*

Proof. We have seen in Lemma 2.1.5 that every Int^c_k is contractible. This ensures the Segal condition, namely that

$$\mathrm{Int}^c_k \xrightarrow{\simeq} \mathrm{Int}^c_1 \overset{h}{\underset{\mathrm{Int}^c_0}{\times}} \cdots \overset{h}{\underset{\mathrm{Int}^c_0}{\times}} \mathrm{Int}^c_1,$$

completeness, and ensures that the given inclusion is a level-wise equivalence. □

2.1.3 The internal category or complete Segal space Int of ordered closed intervals in an open one

We now change our interpretation of the spaces (2.1.3): we do not identify them with the spaces of ordered closed bounded intervals $I_0 \leqslant \cdots \leqslant I_k$ anymore, but as ordered intervals which are closed in (a_0, b_k), i.e. we interpret the elements as

$$\tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k,$$

where $\tilde{I}_j = I_j \cap (a_0, b_k)$ for $0 \leqslant j \leqslant k$. Thus, in the generic case when $a_j \neq a_0$ for $0 < j \leqslant k$ and $b_j \neq b_k$ for $0 \leqslant j < k$, then $\tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k$ are the half-open or closed intervals

$$(a_0, b_0] \leqslant [a_1, b_1] \leqslant \cdots \leqslant [a_{k-1}, b_{k-1}] \leqslant [a_k, b_k).$$

If we view the elements in (2.1.3) in this way, we will denote the internal category (or analogously the Segal space) by Int.

Note that the identity gives an isomorphism of complete Segal spaces describing the change of interpretation:

$$\begin{aligned} \mathrm{Int}_k^c &\longrightarrow \mathrm{Int}_k \\ (I_0 \leqslant \cdots \leqslant I_k) &\longmapsto (\tilde{I}_0 \leqslant \cdots \leqslant \tilde{I}_k), \end{aligned}$$

where $\tilde{I}_j = I_j \cap (a_0, b_k)$ for $j = 0, \ldots, k$. Conversely, $I_j = \mathrm{cl}_{\mathbb{R}}(\tilde{I}_j)$, the closure of $\tilde{I}_j$ in $\mathbb{R}$.

Definition 2.1.8. Let

$$\mathrm{Int}^n_{\bullet,\ldots,\bullet} = (\mathrm{Int}_\bullet)^{\times n}.$$

We denote an element in $\mathrm{Int}^n_{k_1,\ldots,k_n}$ by

$$\underline{I} = (\underline{a}, \underline{b}) = (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1 \leqslant i \leqslant n}.$$

Lemma 2.1.9. *The n-fold simplicial space* $\mathrm{Int}^n_{\bullet,\ldots,\bullet}$ *is a complete n-fold Segal space. Moreover, the inclusion* $* \hookrightarrow \mathrm{Int}^n_{\bullet,\ldots,\bullet}$ *given by degeneracies, where* $*$ *is seen as a constant complete Segal space, is an equivalence of complete n-fold Segal spaces.*

Proof. The Segal condition and completeness follow from the Segal condition and completeness for $\mathrm{Int}_\bullet$. Since every Int_k is contractible by Lemma 2.1.5, $(\mathrm{Int}_\bullet)^{\times n}$ satisfies essential constancy, so Int^n is a complete n-fold Segal space. It also ensures that the given inclusion is a level-wise equivalence. $\square$

2.1.4 The boxing maps

We will need the following maps for convenience later:

Definition 2.1.10. Fix $k \geqslant 0$. The map of spaces

$$\begin{aligned} B : \mathrm{Int}_k &\longrightarrow \mathrm{Int}_0 \\ \underline{I} = (I_0 \leqslant \cdots \leqslant I_k) \to |\Delta^l|_e &\longmapsto B(\underline{I}) = B(\underline{a}, \underline{b}) = (a_0, b_k) \to |\Delta^l|_e \end{aligned}$$

is called the *boxing map*.

Its n-fold product gives, for every $k_1, \ldots, k_n \geqslant 0$, a map $B : \mathrm{Int}^n_{k_1,\ldots,k_n} \to \mathrm{Int}^n_0$ which sends an l-simplex to the (family of) smallest open box(es) containing all intervals,

$$\overline{\underline{I}} = (I^i_0 \leqslant \cdots \leqslant I^i_{k_i})_{1 \leqslant i \leqslant n} \to |\Delta^l|_e \longmapsto B(\overline{\underline{I}}) = B(\overline{\underline{a}}, \overline{\underline{b}}) = (a^1_0, b^1_{k_1}) \times \cdots \times (a^n_0, b^n_{k_n}) \to |\Delta^l|_e.$$

We will usually view the total space of $B(\overline{\underline{I}}) \to |\Delta^l|_e$ as sitting inside $\mathbb{R}^n \times |\Delta^l|_e$ as $\bigcup_{s \in |\Delta^l|_e} B(\overline{\underline{I}}(s)) \times \{s\}$.

We will also require the following rescaling maps.

Definition 2.1.11. For an element $\overline{\underline{I}} \in \mathrm{Int}^n_{k_1,\ldots,k_n}$, let $\rho(\overline{\underline{I}}) : B(\overline{\underline{I}}) \to (0,1)^n$ be the restriction of the product of the affine maps $\mathbb{R} \to \mathbb{R}$ sending a^i_0 to 0 and b^i_k to 1. We call it the *box rescaling map*.

2.1.5 A variant: closed intervals in $(0,1)$

One might prefer to restrict to intervals which lie in $(0,1)$, modifying the definition to

$$\begin{aligned} \mathrm{Int}^{(0,1)}_k = \{(\underline{a}, \underline{b}) = (a_0, \ldots, a_k, b_0, \ldots, b_k) : a_j < b_j \text{ for } 0 \leqslant j \leqslant k, 0 = a_0 \leqslant a_1 \leqslant \cdots \leqslant a_k \\ \text{and } b_0 \leqslant \cdots \leqslant b_{k-1} \leqslant b_k = 1\} \subset \mathrm{Int}_k \end{aligned}$$

The simplicial structure now has to be modified to ensure that the outer endpoints always are 0 and 1. This is provided by composition with an affine rescaling map: Let $g : [m] \to [k]$ be a morphism in Δ. Then, let

$$\begin{aligned} \mathrm{Int}^{(0,1)}_k &\xrightarrow{g^*} \mathrm{Int}^{(0,1)}_m, \\ (I_0 \leqslant \cdots \leqslant I_k) \to |\Delta^l|_e &\longmapsto \rho_g(I_{g(0)} \leqslant \cdots \leqslant I_{g(m)}) \to |\Delta^l|_e, \end{aligned}$$

where the rescaling map $\rho_g = \rho(I_{g(0)} \leqslant \cdots \leqslant I_{g(m)})$ is the unique affine transformation $\mathbb{R} \to \mathbb{R}$ sending $a_{g(0)}$ to 0 and $b_{g(m)}$ to 1.

Lemma 2.1.12. $\mathrm{Int}_\bullet^{(0,1)}$ *is a complete Segal space.*

Proof. The only thing which is not completely analogous to Int^c is checking that it is a simplicial space. Given two maps $[m] \xrightarrow{g} [k] \xrightarrow{\tilde{g}} [p]$, and $I_0 \leqslant \cdots \leqslant I_p$, the rescaling map $\rho_{\tilde{g}\circ g}$ and the composition of the rescaling maps $\rho_{\tilde{g}} \circ \rho_g$ both send $a_{\tilde{g}\circ g(0)}$ to 0 and $b_{\tilde{g}\circ g(m)}$ to 1 and, since affine transformations $\mathbb{R} \to \mathbb{R}$ are uniquely determined by the image of two points, this implies that they coincide. Thus, this gives a functor $\Delta^{op} \to \mathscr{S}pace$. $\square$

Note that the degeneracy maps are the same ones, given by repeating an interval. However, the face maps need to modified: after deleting an end interval we have to rescale the remaining intervals linearly to $(0,1)$. Explicitly, for $j=0$, the rescaling map is the affine map ρ_0 sending $(a_1,1)$ to $(0,1)$, $\rho_0(x) = \frac{x-a_1}{1-a_1}$ and for $j=k$, it is the affine map $\rho_k : (0,b_{k-1}) \to (0,1)$, $\rho_k(x) = \frac{x}{b_{k-1}}$. Then,

$$\begin{array}{ccc} \mathrm{Int}_k^{(0,1)} & \xrightarrow{d_j} & \mathrm{Int}_{k-1}^{(0,1)}, \\ I_0 \leqslant \cdots \leqslant I_k & \longmapsto & \begin{cases} I_0 \leqslant \cdots \leqslant \hat{I}_j \leqslant \cdots \leqslant I_k, & j \neq 0,k, \\ (0, \frac{b_1-a_1}{1-a_1}] \leqslant \cdots \leqslant [\frac{a_k-a_1}{1-a_1}, 1), & j=0, \\ (0, \frac{b_0}{b_{k-1}}] \leqslant \cdots \leqslant [\frac{a_{k-1}}{b_{k-1}}, 1), & j=k. \end{cases} \end{array}$$

Remark 2.1.13. An advantage of this "reduced" version is that the space of objects is just a point: for $k=0$, the condition on the endpoints of the intervals becomes $a_0 = 0$ and $b_0 = 1$, so the only element is $(0,1) \in \mathrm{Int}_0$. In particular, Int_0 is discrete.

Remark 2.1.14. Note that the boxing maps applied to $\mathrm{Int}_k^{(0,1)}$ are trivial: for $\underline{I} = I_0 \leqslant \cdots \leqslant I_k$, we always have that $B(\underline{I}) = (0,1)$. Moreover, $\mathrm{Int}_k^{(0,1)}$ is the preimage of $(0,1)$ under the boxing maps. Finally, note that the simplicial structure is defined exactly as the composition

$$\mathrm{Int}_k^{(0,1)} \xrightarrow{\iota} \mathrm{Int}_k \xrightarrow{g^*} \mathrm{Int}_m \xrightarrow{\rho} \mathrm{Int}_m^{(0,1)},$$

where $\rho : \underline{I} \mapsto (\rho(\underline{I}))(\underline{I})$ consists of applying the box rescaling maps. Moreover, since $\rho \circ \iota = id$, the diagram

$$\begin{tikzcd} \mathrm{Int}_k \arrow[r, shift left, "\rho"] \arrow[d, "g^*"] & \mathrm{Int}_k^{(0,1)} \arrow[l, shift left, hook', "\iota"] \arrow[d, "g^*"] \\ \mathrm{Int}_m \arrow[r, "\rho"] & \mathrm{Int}_m^{(0,1)} \end{tikzcd}$$

commutes and shows that the simplicial structure is defined exactly in a way to ensure that we a natural transformation of simplicial spaces

$$\rho : \mathrm{Int} \longrightarrow \mathrm{Int}^{(0,1)},$$

which is a weak equivalence of complete Segal spaces.

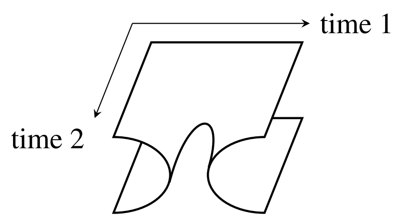
Section 2.2
The (∞,d)-category of n-bordisms

In this section we define an d-fold Segal space $\mathrm{PBord}_n^{(n-d),V}$ in several steps. By applying the Rezk completion functor we obtain a complete d-fold Segal space, the (∞,d)-category of n-bordisms $\mathrm{PBord}_n^{(n-d),V}$. Before we start, we consider the following setup.

Setup. Let V be a finite-dimensional vector space. We first define the levels relative to V with elements being certain submanifolds of the vector space $V\times\mathbb{R}^d \cong V\times B$, where B is an open box, i.e. a product of d bounded open intervals in $\mathbb{R}$. Then we vary V, i.e. we take the limit over all finite-dimensional vector spaces lying in $\mathbb{R}^\infty$. The idea behind this process is that by Whitney's embedding theorem, every manifold can be embedded in some large enough vector space, so in the limit, we include representatives of every n-dimensional manifold. We use $V\times B$ instead of $V\times\mathbb{R}^d$ as in this case the spatial structure is easier to write down explicitly.

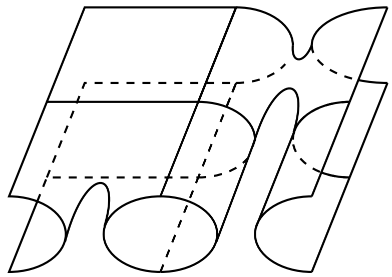
2.2.1 The sets of 0-simplicies of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$

The intuition behind the following definition should be the following. An element (i.e. 0-simplex) in the space $(\mathrm{PBord}_n^{(n-d),V})_{1,\dots,1}$ should be an d-fold bordism, i.e. a manifold for which there are d "time" directions singled out and whose boundary is decomposed into an incoming and an outgoing part in each of these time directions. This is a picture of a simple example for $d=2$.



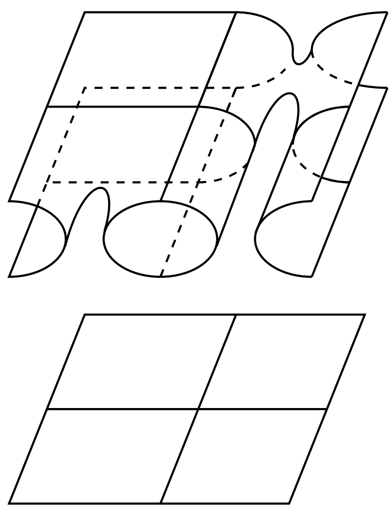
An element in the space $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ should be an d-fold bordism, which is the composition of k_1 bordisms in the first "time" direction, k_2 bordisms in the second "time" direction, and so on.

This is a picture of an example for $n = 2$, $d = 2$ and $k_1 = k_2 = 2$.

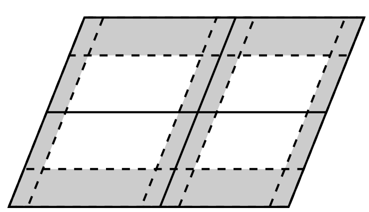


The pictures both depict the bordisms as embedded into $\mathbb{R}$ times the two time directions. We would like to point out that the "time" directions have a preferred ordering, as we will discuss in more detail later.

More generally, we will choose the bordisms to be equipped with an embedding into some finite dimensional real vector space V times d "time" directions, which we single out to track where the bordism is allowed to be cut into the individual composed bordisms. Furthermore, to keep track of the "cuts", we need to remember the data of the grid in the "time" directions.



In practice, we will keep track of little intervals surrounding the grid instead of the grid itself. This should be thought of as remembering little collars around the cuts rather than the cuts themselves.



We will explain how to recover the cuts and how to interpret the following definition in the example and remark right after the definition.

Notation 2.2.1. For $S \subseteq \{1, \dots, d\}$, denote the projection from $\mathbb{R}^d$ onto the coordinates indexed by S by $\pi_S : \mathbb{R}^d \to \mathbb{R}^S$.

We will now define the sets of 0-simplices of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ and denote them by $(\mathbb{P}\mathrm{Bord}_n^{(n-d),V})_{k_1,\dots,k_d}$ to avoid adding an extra index.

Definition 2.2.2. Let V be a finite-dimensional $\mathbb{R}$-vector space, which we identify with some $\mathbb{R}^r$. For every d-tuple $k_1, \dots, k_d \geqslant 0$, let $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ be the collection of tuples

$$(M, \underline{\bar{I}} = (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1 \leqslant i \leqslant d}),$$

satisfying the following conditions:

(1) For $1 \leqslant i \leqslant d$,
$$(I_0^i \leqslant \cdots \leqslant I_{k_i}^i) \in \mathrm{Int}_{k_i}.$$

(2) M is a closed and bounded n-dimensional submanifold of $V \times B(\underline{\bar{I}})$ and the composition $\pi : M \hookrightarrow V \times B(\underline{\bar{I}}) \twoheadrightarrow B(\underline{\bar{I}})$ is a proper map.[b]

(3) For every $S \subseteq \{1, \dots, d\}$, let $p_S : M \xrightarrow{\pi} B(\underline{\bar{I}}) \xrightarrow{\pi_S} \mathbb{R}^S$ be the composition of π with the projection π_S onto the S-coordinates. Then for every $1 \leqslant i \leqslant d$ and $0 \leqslant j_i \leqslant k_i$, at every $x \in p_{\{i\}}^{-1}(I_{j_i}^i)$, the map $p_{\{i,\dots,d\}}$ is submersive.

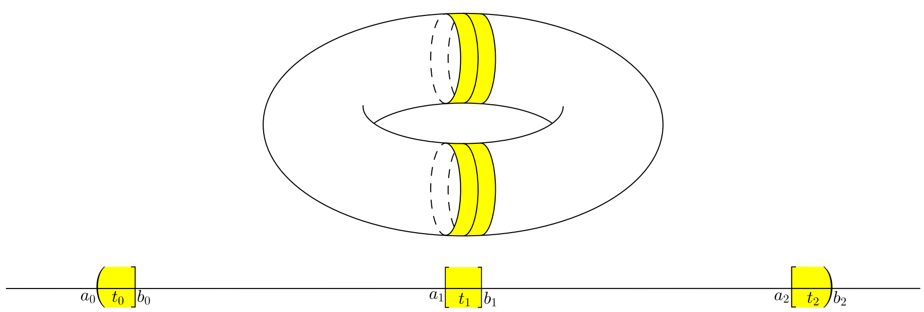


Figure 2.1: An element of $(\mathbb{P}\mathrm{Bord}_2^{1,V})_2$

[b]Recall the boxing map from Section 2.1.4.

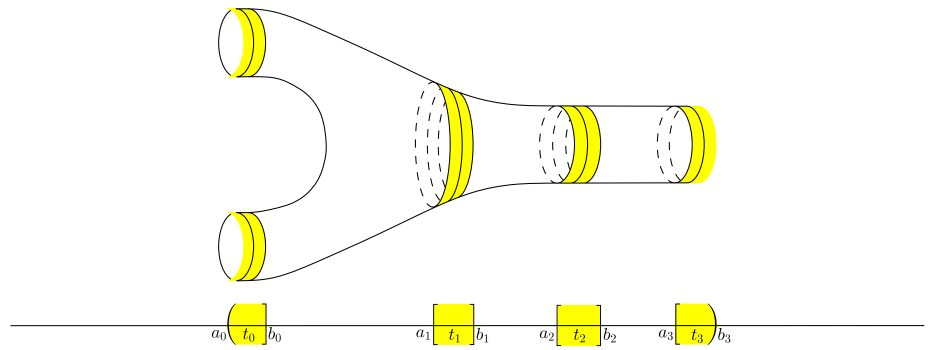
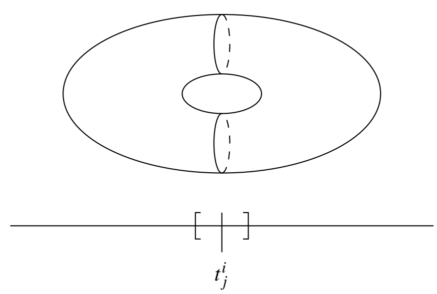


Figure 2.2: An element of $\left(\mathbb{P}\mathrm{Bord}_2^{1,V}\right)_3$

Remark 2.2.3. For $k_1, \ldots, k_d \geqslant 0$, one should think of an element in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ as a collection of $k_1 \cdots k_d$ composed bordisms, with k_i composed bordisms with collars in the ith direction. They can be understood as follows.

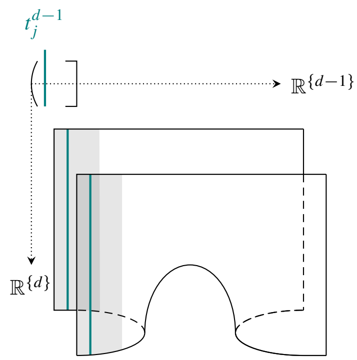
- Condition (3) in particular implies that for every $1 \leqslant i \leqslant d$, at every $x \in p_{\{i\}}^{-1}(I_j^i)$, the map $p_{\{i\}}$ is submersive. So if we choose $t_j^i \in I_j^i$, it is a regular value of $p_{\{i\}}$, and therefore $p_{\{i\}}^{-1}(t_j^i)$ is an $(n-1)$-dimensional manifold. The embedded manifold M should be thought of as a composition of n-bordisms and $p_{\{i\}}^{-1}(t_j^i)$ is one of the $(n-1)$-bordisms (or a composition therof) in the composition.



- For any $t_j^{d-1} \in I_j^{d-1}$ and $t_l^{d-1} \in I_l^{d-1}$, there is an inclusion of the preimages

$$p_{\{d-1,d\}}^{-1}\left((t_j^{d-1}, t_l^d)\right) \subset p_{\{d-1\}}^{-1}(I_j^{d-1}),$$

  and by condition (3) the map $p_{\{d-1,d\}}$ is submersive there. Therefore $p_{\{d-1,d\}}^{-1}\left((t_j^{d-1}, t_l^d)\right)$ is an $(n-2)$-dimensional manifold, which should be thought of as one of the $(n-2)$-bordisms which are connected by the composition of n-bordisms M. Moreover, again since $p_{\{d-1,d\}}$ is submersive everywhere in $p_{\{d-1\}}^{-1}(I_j^{d-1})$, a variant of Ehresmann's fibration theorem shows that the preimage $p^{-1}_{\{d-1\}}(t^{d-1}_j)$ is a trivial fibration and thus a trivial $(n-1)$-bordism between the $(n-2)$-bordisms it connects.



- Similarly, for $(t^k_{j_k}, \ldots, t^d_{j_d}) \in I^k_{j_k} \times \cdots \times I^d_{j_d}$, the preimage

$$p^{-1}_{\{k,\ldots,d\}}\left((t^k_{j_k}, \ldots, t^d_{j_d})\right)$$

  is a $(k-1)$-dimensional manifold, which should be thought of as one of the $(k-1)$-bordisms which is connected by the composition of n-bordisms M.

2.2.2 Construction of the topological space $(\mathfrak{PBord}^{(n-d),V}_n)_k$

Definition 2.2.4. Let $\Psi(V \times (0,1)^d)$ be the set of subsets $M \subseteq V \times (0,1)^d$ which are smooth, bounded n-dimensional submanifolds without boundary, and such that M is closed as a subset.

Step 1. We first define the *compactly supported topology* on $\Psi(V \times (0,1)^d)$. We will write $\Psi(V \times (0,1)^d)^{cs}$ for $\Psi(V \times (0,1)^d)$ equipped with this topology. In fact, $\Psi(V \times (0,1)^d)^{cs}$ will be an infinite-dimensional smooth manifold, in which a neighbourhood of $M \in \Psi(V \times (0,1)^d)^{cs}$ is homeomorphic to a neighbourhood of the zero-section in the vector space $\Gamma_c(NM)$ consisting of compactly supported sections of the normal bundle NM of $M \subseteq V \times (0,1)^d$.

Construction 2.2.5. Let $C^\infty_c(M)$ denote the set of compactly supported smooth functions on M. Given a function $\varepsilon : M \to (0,\infty)$ and finitely many vector fields $X = (X_1, X_2, \ldots, X_r)$ on M, let $B(\varepsilon, X)$ denote the set of all functions such that $|(X_1X_2 \ldots X_r f)(x)| < \varepsilon(x)$ for all x. Declare the family of sets of the form $f + B(\varepsilon, X)$ a subbasis for the topology on $C^\infty_c(M)$, as f ranges over $C^\infty_c(M)$, ε over functions $M \to (0,\infty)$, and X over r-tuples of vector fields, and r over non-negative integers. This makes $C^\infty_c(M)$ into a locally convex vector space.

We define the normal bundle NM to be the subbundle of ε^n which is the orthogonal complement to the tangent bundle $TM \subseteq \varepsilon^n$. This identifies $\Gamma_c(NM)$ with a linear subspace of $C^\infty_c(M)^{\oplus n}$. We topologise it as a subspace.

By the tubular neighbourhood theorem, the standard map $NM \to \mathbb{R}^d$ restricts to an embedding of a neighbourhood of the zero section. Associating to a section s its image $s(M)$ gives a partially defined injective map

$$\Gamma_c(NM) \overset{c_M}{\dashrightarrow} \Psi(V \times (0,1)^d)^{cs}$$

whose domain is an open set. Topologise $\Psi(V \times (0,1)^d)^{cs}$ by declaring the maps c_M to be homeomorphisms onto open sets. This makes $\Psi(V \times (0,1)^d)^{cs}$ into an infinite dimensional manifold, modelled on the topological vector space $\Gamma_c(NM)$.

Step 2. For each compact set $K \subseteq V \times (0,1)^d$, we define a topology on $\Psi(V \times (0,1)^d)$, called the *K-topology*. We will write $\Psi(V \times (0,1)^d)^K$ for $\Psi(V \times (0,1)^d)$ equipped with this topology.

Construction 2.2.6. Let

$$\Psi(U)^{cs} \xrightarrow{\pi_K} \Psi(K \subseteq U)$$

be the quotient map that identifies elements of $\Psi(V \times (0,1)^d)^{cs}$ if they agree on a neighbourhood of K. The image of a manifold $M \in \Psi(V \times (0,1)^d)^{cs}$ is the germ of M near K, and we shall also write $\pi_K(M) = M|_K$. Give $\Psi(K \subseteq V \times (0,1)^d)$ the quotient topology.

Now, let $\Psi(V \times (0,1)^d)^K$ be the topological space with the same underlying set as $\Psi(V \times (0,1)^d)^{cs}$, and with the coarsest topology making $\pi_K : \Psi(V \times (0,1)^d)^K \to \Psi(K \subseteq V \times (0,1)^d)$ continuous. It is a formal consequence of the universal properties of initial and quotient topologies that the identity map $\Psi(V \times (0,1)^d)^L \to \Psi(V \times (0,1)^d)^K$ is continuous when $K \subseteq L$ are two compact sets. That is, the L-topology is finer than the K-topology.

Step 3. Finally, let $\Psi(V \times (0,1)^d)$ have the coarsest topology finer than all the K-topologies. In other words, $\Psi(V \times (0,1)^d)$ is the inverse limit of $\Psi(V \times (0,1)^d)^K$ over larger and larger compact sets.

Now, we identify the topology on $\Psi(V \times (0,1)^d)$ with the quotient

$$\mathrm{Sub}(V \times (0,1)^d) \xleftarrow{\simeq} \coprod_{[M]} \mathrm{Emb}(M, V \times (0,1)^d)/\mathrm{Diff}(M),$$

where the coproduct is taken over diffeomorphism classes of n-manifolds. It is given by defining the neighborhood basis at M to be

$$\{N \subset V \times (0,1)^d : N \cap K = j(M) \cap K, j \in W\},$$

where $K \subset V \times (0,1)^d$ is compact and $W \subseteq \mathrm{Emb}(M, V \times (0,1)^d)$ is a neighborhood of the inclusion $M \hookrightarrow V \times (0,1)^d$ in the Whitney C^∞-topology. Thus we obtain a topology on

$$\mathrm{Sub}(V \times (0,1)^d) \times \mathfrak{Int}^d_{k_1,\ldots,k_d},$$

where we view $\mathfrak{Int}^d_{k_1,\dots,k_d}$ as a (topological) subspace of $\mathbb{R}^{2k}$ as in 2.1.1.

For an element $\underline{I} \in \mathfrak{Int}^d_{k_1,\dots,k_d}$, recall from Definition 2.1.11 the box rescaling map $\rho(\underline{I}) : B(\underline{I}) \to (0,1)^d$. Then we identify an element $(M,\underline{I}) \in (\mathfrak{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ whose underlying submanifold is the image of an embedding $\iota : M \hookrightarrow V \times B(\underline{I})$ with the element $\big([\rho(\underline{I}) \circ \iota], \rho(\underline{I})\big)$ in the above space. This identification gives an inclusion

$$(\mathfrak{PBord}_n^{(n-d),V})_{k_1,\dots,k_d} \subseteq \mathrm{Sub}(V \times (0,1)^d) \times \mathfrak{Int}^d_{k_1,\dots,k_d},$$

which we use to topologize the left-hand side.

The Kan complex $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$

To model the levels of the bordism category as spaces, i.e. as Kan complexes, we can start with the above version as a topological space and take singular simplices of this topological space. However, smooth maps from a smooth manifold X to $\Psi(V \times (0,1)^d)$ are easier to handle. By Lemma 2.18 [GRW10], every continuous map from a smooth manifold, in particular from $|\Delta^l|_e$, to $(\mathfrak{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ can be perturbed to a smooth one, so the homotopy type when considering smooth singular simplices does not change.

We will first give a very explicit description of the Kan complex $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$.

Definition 2.2.7. An l-simplex of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ consists of tuples

$$(M, \underline{I}(s) = (I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{s\in|\Delta^l|_e}$$

such that

(1) $\underline{I} = (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{1\leqslant i\leqslant d} \to |\Delta^l|_e$ is an l-simplex in $\mathrm{Int}^d_{k_1,\dots,k_d}$,

(2) M is a closed and bounded $(n+l)$-dimensional submanifold of $V \times B(\underline{I}(s))_{s\in|\Delta^l|_e} \subset V \times \mathbb{R}^d \times |\Delta^l|_e$ such that[c]

 (a) the composition $\pi : M \hookrightarrow V \times B(\underline{I}(s))_{s\in|\Delta^l|_e} \twoheadrightarrow B(\underline{I}(s))_{s\in|\Delta^l|_e}$ of the inclusion with the projection is proper,

 (b) its composition with the projection onto $|\Delta^l|_e$ is a submersion $M \to |\Delta^l|_e$ which is trivial outside $|\Delta^l| \subset |\Delta^l|_e$, and

(3) for every $S \subseteq \{1,\dots,d\}$, let $p_S : M \xrightarrow{\pi} B(\underline{I}(s))_{s\in|\Delta^l|_e} \subset \mathbb{R}^d \times |\Delta^l|_e \xrightarrow{\pi_S} \mathbb{R}^S \times |\Delta^l|_e$ be the composition of π with the projection π_S onto the S-coordinates. Then for every $1 \leqslant i \leqslant d$ and $0 \leqslant j_i \leqslant k_i$, at every $x \in p_{\{i\}}^{-1}(\bigcup_{s\in|\Delta^l|_e} I^i_{j_i}(s) \times \{s\})$, the map $p_{\{i,\dots,d\}}$ is submersive.

[c]Recall that we view the total space of $B(\underline{I}) \to |\Delta^l|_e$ as sitting inside $\mathbb{R}^d \times |\Delta^l|_e$ as $\bigcup_{s\in|\Delta^l|_e} B(\underline{I}(s)) \times \{s\}$.

From the Definition of smooth map in [GRW10, Definition 2.16, Lemma 2.17] we immediately get:

Lemma 2.2.8. *An l-simplex of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ is exactly a smooth l-simplex of $(\mathfrak{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$.*

Remark 2.2.9. Note that for $l = 0$ we recover Definition 2.2.2. Moreover, for every $s \in |\Delta^l|_e$ the fiber M_s of $M \to |\Delta^l|_e$ determines an element in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$

$$(M_s) = (M_s \subset V \times B(\underline{I}(s)), \underline{I}(s)).$$

We will use the notation $\pi_s : M_s \to B(\underline{I}(s))$ for the composition of the embedding and the projection.

Remark 2.2.10. The conditions ((2)a), ((2)b), and ((3)) imply that $M \to |\Delta^l|_e$ is a smooth fiber bundle, and, since $|\Delta^l|_e$ is contractible, even a trivial fiber bundle. The proof is a more elaborate version of the argument after Definition 2.6 in [GTMW09].

We now use the simplicial maps of the space $\mathrm{Int}^d_{k_1,\ldots,k_d}$ to explain those of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$.

Definition 2.2.11. Fix $k \geqslant 0$ and let $f : [m] \to [l]$ be a morphism in the simplex category Δ. Then let $|f| : |\Delta^m|_e \to |\Delta^l|_e$ be the induced map between standard simplices.

Let f^Δ be the map sending an l-simplex in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ to the m-simplex which consists of

(1) for $1 \leqslant i \leqslant d$, the m-simplex in Int_{k_i} obtained by applying f^Δ,

$$f^\Delta\Big((I_0^i(s) \leqslant \cdots \leqslant I_{k_i}^i(s))_{s\in|\Delta^l|_e}\Big) = \big(I_0^i(|f|(s)) \leqslant \ldots \leqslant I_{k_i}^i(|f|(s))\big)_{s\in|\Delta^m|_e},$$

(2) The $(n+m)$-dimensional submanifold $f^\Delta M \subseteq V \times B(\underline{I}(s))_{s\in|\Delta^m|_e}$ obtained by the pullback of $M \to |\Delta^l|_e$ along $|f|$. Note that its fiber at $s \in |\Delta^m|_e$ is $(f^\Delta M)_s = M_{|f|(s)}$ and

$$f^\Delta M = \bigcup_{s\in|\Delta^m|_e} M_{|f|(s)} \times \{s\}.$$

The above assignment is indeed well-defined since the underlying assignment for the underlying intervals is well-defined and since the map $|f|$ is a submersion, the pullback of $M \to |\Delta^l|_e$ along $|f|$ is also a submersion. Moreover, the assignment is functorial, since pullback commutes contravariantly with composition, and thus $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ is a simplicial set.

Proposition 2.2.12. $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ *is the smooth singular space of* $(\mathfrak{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$. *In particular, it is a space.*

Proof. By definition the simplicial maps f^Δ are induced precisely by the maps $|f| : |\Delta^m|_e \to |\Delta^l|_e$. $\square$

Notation 2.2.13. We denote the *spatial face and degeneracy maps* of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ by d_j^Δ and s_j^Δ for $0 \leqslant j \leqslant l$.

Example 2.2.14. We now construct an example of a path. It shows that cutting off part of the collar of a bordism yields an element which is connected to the original one by a path.

Let $(M) = (M, \underline{I} = (I_0^i \leqslant \cdots \leqslant I_{k_i}^i)_{i=1,\dots,d}) \in (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ and fix $1 \leqslant i \leqslant d$. We show that cutting off a short enough piece in the ith direction at an end of an element of $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ leads to an element which is connected by a path to the original one. Fix $1 \leqslant i \leqslant d$ and let $\varepsilon < b_0^i - a_0^i$.

Choose a smooth, increasing, bijective function $[0,1] \to [0,\varepsilon], s \mapsto \varepsilon(s)$ with vanishing derivative at the endpoints.

For $0 \leqslant j \leqslant k_i$ and $s \in [0,1] \subset |\Delta^1|_e$ let

$$I_j^i(s) = (a_0^i + \varepsilon(s), b_{k_i}^i) \cap I_j^i,$$

and then $B(\underline{I}(s)) = (a_0^i + \varepsilon(s), b_{k_i}^i) \subset B(\underline{I})$. For $s \leqslant 0$ and $s \geqslant 1$ let the family be constant. Then let $M(\varepsilon)$ be the preimage of the subset $\bigcup_{s \in |\Delta^1|_e} B(\underline{I}(s)) \times \{s\} \subseteq B(\underline{I}) \times |\Delta^1|_e$ of $M \times |\Delta^1|_e \to B(\underline{I}) \times |\Delta^1|_e$, i.e. the submanifold

$$\begin{tikzcd}
M(\varepsilon) \arrow[r, hook] \arrow[d] & M \times |\Delta^1|_e \arrow[d] \\
\bigcup_{s \in |\Delta^1|_e} B(\underline{I}(s)) \times \{s\} \arrow[r, hook] & B(\underline{I}) \times |\Delta^1|_e
\end{tikzcd}$$

Then $(M(\varepsilon), \underline{I}(s))$ is a 1-simplex in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ with fibers $M(\varepsilon)_s = p_{\{i\}}^{-1}\big((a_0^i + \varepsilon(s), b_k^i)\big)$.

Remark 2.2.15. In the above example we constructed a path from an element in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ to its *cutoff*, where we cut off the preimage of $p_i^{-1}((a_0^i, \varepsilon])$ for suitably small ε. Note that the same argument holds for cutting off the preimage of $p_i^{-1}([b_{k_i}^i - \delta, b_{k_i}^i))$ for suitably small δ. Moreover, we can iterate the process and cut off ε_i, δ_i strips in all i directions. Choosing $\varepsilon_i = \frac{b_0^i - a_0^i}{2}, \delta_i = \frac{b_{k_i}^i - a_{k_i}^i}{2}$ yields a path to its *cutoff* with underlying submanifold

$$cut(M) = \pi^{-1}\Big(\prod_{i=1}^d \big(\frac{a_0^i + b_0^i}{2}, \frac{a_{k_i}^i + b_{k_i}^i}{2}\big)\Big).$$

2.2.3 The n-fold simplicial space $(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\cdots,\bullet}$

We make the collection of spaces $(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\ldots,\bullet}$ into an d-fold simplicial space by lifting the simplicial structure of $\mathrm{Int}^{\times d}_{\bullet,\ldots,\bullet}$. We first need to extend the assignment

$$([k_1],\ldots,[k_d]) \longmapsto (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$$

to a functor from $(\Delta^{\mathrm{op}})^d$.

Definition 2.2.16. For every $1 \leqslant i \leqslant d$, let $g_i : [m_i] \to [k_i]$ be a morphism in Δ, and denote by $g = (g_i)_i$ their product in Δ^d. Then

$$(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d} \xrightarrow{g^*} (\mathrm{PBord}_n^{(n-d),V})_{m_1,\ldots,m_d}.$$

applies g_i^* to the ith tuple of intervals and perhaps cuts the manifold. Explicitly, on l-simplices, g^* sends an element

$$(M \subset V \times B(\underline{I}(s))_{s\in|\Delta^l|_e}, \underline{I}(s) = (I_0^i(s) \leqslant \ldots \leqslant I_{k_i}^i(s))_{i=1}^d)$$

to

$$\big(g^*M = \pi^{-1}\big(B(g^*\underline{I}(s))_{s\in|\Delta^l|_e}\big) \subset V \times B(\underline{I}(s))_{s\in|\Delta^l|_e}, g^*(\underline{I})(s) = (I^i_{g(0)}(s) \leqslant \ldots \leqslant I^i_{g(m_i)}(s))_{i=1}^d)\big),$$

where $\pi : M \subset V \times B(\underline{I}(s))_{s\in|\Delta^l|_e} \twoheadrightarrow B(\underline{I}(s))_{s\in|\Delta^l|_e}$. Note that $(g^*M)_s = g^*M_s$.

Note that as the manifold g^*M is the preimage of the new box, we just cut off the part of the manifold outside the new box. This is functorial, as it is functorial on the intervals, and, if $\tilde{g}_i : [k_i] \to [\tilde{k}_i]$ and $\tilde{g} = (\tilde{g}_i)_i$, the following diagram commutes by construction:

$$\begin{tikzcd}
M \arrow[d, "\pi"] & \supseteq & g^*M \arrow[d, "\pi"] & \supseteq & \tilde{g}^*g^*M \arrow[d, "\pi"] \\
B(\underline{I}(s))_{s\in|\Delta^l|_e} & \supseteq & B(g^*(\underline{I}(s))_{s\in|\Delta^l|_e} & \supseteq & B(\tilde{g}^*g^*(\underline{I}(s))_{s\in|\Delta^l|_e}
\end{tikzcd}$$

Notation 2.2.17. We denote the *(simplicial) face and degeneracy maps* by $d_j^i : (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d} \to (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_i-1,\ldots,k_d}$ and $s_j^i : (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d} \to (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_i+1,\ldots,k_d}$ for $0 \leqslant j \leqslant k_i$.

Notation 2.2.18. Recall from remark 2.2.3 that for $k_1,\ldots,k_d \geqslant 0$, one should think a 0-simplex in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d}$ as a collection of $k_1\cdots k_d$ composed bordisms with k_i composed bordisms with collars in the ith direction. These composed collared bordisms are the images under the maps

$$D(j_1,\ldots,j_d) : (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_d} \longrightarrow (\mathrm{PBord}_n^{(n-d),V})_{1,\ldots,1}$$

for $(1 \leqslant j_i \leqslant k_i)_{1 \leqslant i \leqslant d}$ arising as compositions of inert face maps, i.e. $D(j_1, \dots, j_d)$ is the map determined by the maps

$$d(j_i) : [1] \to [k_i], \quad (0 < 1) \mapsto (j_i - 1 < j_i)$$

in the category Δ. This should be thought of as sending an element to the $(j_1, \dots, j_d)$-th collared bordism in the composition. Moreover, we will later use the notation

$$D^i(j_i) : (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d} \longrightarrow (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,1,\dots,k_d}$$

for the maps induced by just $d(j_i)$. By abuse of notation, we will denote the submanifold $d(j_i)^*M$ by $D^i(j_i)(M)$.

Proposition 2.2.19. *The spatial and simplicial structures of $(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet}$ are compatible, i.e. for $f : [l] \to [p]$, $g_i : [m_i] \to [k_i]$ for $1 \leqslant i \leqslant d$, the induced maps*

$$f^\Delta \text{ and } g^*$$

commute. We thus obtain an d-fold simplicial space $(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\cdots,\bullet}$.

Proof. Since Int^d is a simplicial space, it is enough to show that the maps commute on the manifold part, i.e.

$$g^* f^\Delta M = f^\Delta g^* M.$$

This follows from the commuting of the following diagram, in which all sides arise from taking preimages. The preimages are taken over $B(g^*\underline{I}(s))_{s \in |\Delta^m|_e} \subset B(\underline{I}(s))_{s \in |\Delta^m|_e}$ and $|f| : |\Delta^m|_e \to |\Delta^l|_e$, respectively, which affect different components of $V \times \bigcup_{s \in |\Delta^m|_e} (B(\underline{I}(s)) \times \{s\}) \subset V \times \mathbb{R}^d \times |\Delta^m|_e$, so they commute.

$$\begin{tikzcd}
 & V \times B(\underline{I}(s))_{s \in |\Delta^m|_e} \arrow[dd, "id \times |f|"] & & V \times B(g^*\underline{I}(s))_{s \in |\Delta^m|_e} \arrow[ll, hook'] \arrow[dd, "id \times |f|"] \\
f^\Delta M \arrow[ur, hook, dashed] \arrow[dd, dashed] & & g^* f^\Delta M = f^\Delta g^* M \arrow[ll, dashed] \arrow[ur, hook, dashed] \arrow[dd, dashed] & \\
 & V \times B(\underline{I}(s))_{s \in |\Delta^l|_e} & & V \times B(g^*\underline{I}(s))_{s \in |\Delta^l|_e} \arrow[ll, hook'] \\
M \arrow[ur, hook] & & g^*M \arrow[ll, dashed] \arrow[ur, hook, dashed] &
\end{tikzcd}$$

□

2.2.4 The complete n-fold Segal space Bord_n

We will now prove that $\mathrm{PBord}_n^{(n-d),V}$ leads to a pre-(∞,d)-category, i.e. a complete d-fold Segal space of bordisms.

Proposition 2.2.20. $(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet}$ *is an d-fold Segal space.*

Proof. We need to prove that the Segal condition and globularity are satisfied.

The Segal condition is satisfied. Fix fixed $k_1,\dots,k_d \geqslant 0$. We need to show that for every $1 \leqslant i \leqslant d$, and $k_i = m+l$, the Segal map

$$\gamma_{m,l} : (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_i,\dots,k_n} \longrightarrow (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,m,\dots,k_d} \underset{(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,0,\dots,k_d}}{\overset{h}{\times}} (\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,l,\dots,k_d}$$

is a weak equivalence. From now on we will often omit writing out the indices for $\alpha \neq i$ for clarity.

Since every level set $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\dots,k_d}$ is a Kan complex by proposition 2.2.12, i.e. fibrant, the homotopy fiber product on the right hand side can be chosen to be the space of triples consisting of two points and a path between their target and source, respectively.

Note that an element in this space is given by a triple consisting of

$$\begin{aligned}(M,\underline{I}) &= (\iota : M \subset V \times B(\underline{I}), \underline{I} = \left(I_0^i \leqslant \cdots \leqslant I_m^i, I_0^j \leqslant \cdots \leqslant I_{k_j}^j\right)_{1\leqslant j\leqslant d, j\neq i}),\\ (N,\underline{J}) &= (\kappa : N \subset V \times B(\underline{J}), \underline{J} = \left(J_0^i \leqslant \cdots \leqslant J_l^i, J_0^j \leqslant \cdots \leqslant J_{k_j}^j\right)_{1\leqslant j\leqslant d, j\neq i}),\end{aligned}$$

together with a path h from the target $D^i(m)(M,\underline{I}) = \left(D^i(m)(M), I_m^i, (I_0^j \leqslant \cdots \leqslant I_{k_j}^j)_{1\leqslant j\leqslant d, j\neq i}\right)$ of $(M,\underline{I})$ in the ith direction to the source $D^i(1)(N,\underline{J}) = \left(D^i(1)(N), J_0^i, (J_0^j \leqslant \cdots \leqslant J_{k_j}^j)_{1\leqslant j\leqslant d, j\neq i}\right)$ of $(N,\underline{J})$ in the ith direction (using Notation 2.2.18).

The Segal map $\gamma_{m,l}$ factors as a composition

$$\begin{tikzcd} (\mathrm{PBord}_n^{(n-d),V})_{k_i} \arrow[r, "\gamma_{m,l}"] \arrow[d] & (\mathrm{PBord}_n^{(n-d),V})_m \underset{(\mathrm{PBord}_n^{(n-d),V})_0}{\overset{h}{\times}} (\mathrm{PBord}_n^{(n-d),V})_l \\ (\mathrm{PBord}_n^{(n-d),V})^{m,l} \arrow[r, hook] & \mathrm{P}_{k_i}^{m,l}, \arrow[u] \end{tikzcd} \tag{2.2.1}$$

as follows: Informally, the lower right hand corner is the subspace of triples for which, for the directions besides the ith, the tuples of intervals agree and the path of intervals is constant. The lower left hand corner is the subspace thereof, for which in addition in the ith direction $I^i_m = J^i_0$, and along the path this interval stays constant. We will define these spaces below. Our strategy to prove that $\gamma_{m,l}$ is a weak equivalence is to show that all three maps are weak equivalences. Here the left vertical map is the main step of the proof – this is where "composing" the bordisms happens, as we will see below. That the bottom and right vertical map are weak equivalences follows from a rescaling procedure. Let us first define the two spaces in question.

For the lower right hand corner, for $1 \leqslant j \leqslant d$ and $j \neq i$, consider the jth forgetful map

$$\mathrm{PBord}_n^{(n-d),V} \longrightarrow \mathrm{Int}, \quad (M, \underline{I}) \longmapsto \underline{I}^j.$$

The canonical maps from the pullback to the homotopy pullback $\mathrm{Int}_\bullet \cong \mathrm{Int}_\bullet \underset{\mathrm{Int}_\bullet}{\times} \mathrm{Int}_\bullet \to \mathrm{Int}_\bullet \overset{h}{\underset{\mathrm{Int}_\bullet}{\times}} \mathrm{Int}_\bullet$ (which is a weak equivalence since a deformation retract is straightforward to write down and rescales the second tuple of intervals) for varying j induce a (strict) pullback square

$$\begin{tikzcd}
(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet,m,\bullet,\dots,\bullet} \overset{h}{\underset{(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet,0,\bullet,\dots,\bullet}}{\times}} (\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet,l,\bullet,\dots,\bullet} \arrow[r] & \mathrm{Int}_{\bullet,\dots,\bullet}^{\times(n-1)} \overset{h}{\underset{\mathrm{Int}_{\bullet,\dots,\bullet}^{\times(n-1)}}{\times}} \mathrm{Int}_{\bullet,\dots,\bullet}^{\times(n-1)} \\
P^{m,l}_{\bullet,\dots,\bullet} \arrow[u] \arrow[r] & \mathrm{Int}_{\bullet,\dots,\bullet}^{\times(n-1)}. \arrow[u, "\simeq"']
\end{tikzcd}$$

The strict pullback of this diagram consists of exactly those pairs whose jth tuples of intervals agree for every $j \neq i$, and is constant along the path (but the embedded manifold can still vary).[d]

For the lower left hand corner, consider the canonical map $\mathrm{Int}_m \underset{\mathrm{Int}_0}{\times} \mathrm{Int}_l \to \mathrm{Int}_m \overset{h}{\underset{\mathrm{Int}_0}{\times}} \mathrm{Int}_l$ (which is a weak equivalence since both sides are contractible). Now form the (strict) pullback

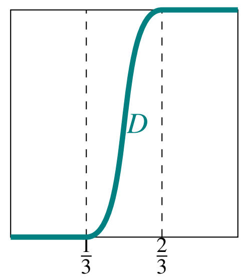
$$\begin{tikzcd}
P^{m,l}_{\bullet,\dots,\bullet} \arrow[r] & \mathrm{Int}_m \overset{h}{\underset{\mathrm{Int}_0}{\times}} \mathrm{Int}_l \\
(\mathrm{PBord}_n^{(n-d),V})^{m,l}_{\bullet,\dots,\bullet} \arrow[u] \arrow[r] & \mathrm{Int}_m \underset{\mathrm{Int}_0}{\times} \mathrm{Int}_l. \arrow[u, "\simeq"']
\end{tikzcd}$$

It consists of exactly those pairs whose jth tuples of intervals agree for every $j \neq i$ and is constant along the path (but the embedded manifold can still vary), and, in addition, in the ith direction, the last interval of the first element is the first interval of the second element.[e]

[d]Note that since the right vertical map is a weak equivalence, if the diagram were also a homotopy pullback diagram, we would immediately see that the left vertical map is a weak equivalence as well. However, neither map in the diagram is a fibration (or not even a "sharp map" à la Rezk [Rez98]), so we need to find a different strategy.

[e]Again, the right vertical map is a weak equivalence, and it would be more convenient to take the homotopy pullback. However, the same problem appears as in the previous situation.

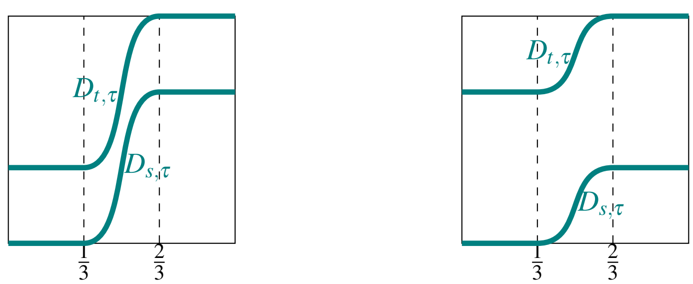
The left vertical map in (2.2.1) *is a weak equivalence:* We first fix once and for all a "smoothed diagonal" $D \subset [0,1]^2$: it is the graph of a map $\varsigma : [0,1] \to [0,1]$, which has vanishing derivative in $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ (we could also chose fixed shorter intervals) and is bijective with smooth inverse in $[\frac{1}{3}, \frac{2}{3}]$, for example



We will use this to define a deformation retract of $\gamma^{m,l}$ which we suggestively call *glue*. The homotopy exhibiting the deformation retract will use the following two modified functions for $\tau \in [0,1]$. Let

$$\varsigma^s_\tau = \tau \cdot \varsigma \quad \text{and} \quad \varsigma^t_\tau = 1 + \tau \cdot (\varsigma - 1).$$

Then for $\tau = 1$ we have that $\varsigma = \varsigma^s_1 = \varsigma^t_1$, and for $\tau = 0$ we have $\varsigma^s_0 = 0$ and $\varsigma^t_0 = 1$. Moreover, for every τ, both ς^s_τ and ς^t_τ are smooth and bijective onto its image. These give "flatter" diagonals $D_{s,\tau}$ and $D_{t,\tau}$.



$\tau = \frac{1}{3}$ $\qquad$ $\tau = \frac{2}{3}$

Recall from above that an element in $(\mathrm{PBord}_n^{(n-d),V})^{m,l}_{\bullet,\dots,\bullet}$ is given by a pair $(M, \underline{I})$ and $(N, \underline{J})$ and a path h from the target of the former to the source of the latter, along which the interval is constant. We will use this path h to glue the embedded manifolds M and N. A similar argument works for l-simplices in $(\mathrm{PBord}_n^{(n-d),V})^{m,l}_{\bullet,\dots,\bullet}$.

The 1-simplex h by definition is a submanifold P of[f] $V \times (c,b) \times |\Delta^1|_e$ such that the composition with the projection $\pi_{\{i\}} : P \to (c,b) \times |\Delta^1|_e$ is a submersion. We rescale the fixed smoothed diagonal D linearly to obtain a smooth diagonal $D^{c,b}$ in $(c,b) \times |\Delta^1|_e$.

[f]Actually, of $V \times (c,b) \times B((I_0^j \leqslant \dots \leqslant I_{k_j}^j)_{1 \leqslant j \leqslant d, j \neq i}) \times |\Delta^1|_e$.

Consider the preimage P_{diag} of $\pi_{\{i\}}$ of $D^{c,b}$. Since the projection $\pi_{\{i\}} : P \to (c,b)\times|\Delta^1|_e$ is submersive, a Morse lemma style argument shows that this preimage P_{diag} is diffeomorphic to both $D(m)(M)$ and $D(1)(N)$. Thus we glue the manifolds M and N over P_{diag} to obtain $M\cup_{P_{diag}}N$. We realize it as a submanifold of $V\times\mathbb{R}\times(a,d)$ by using

$$M\cong M\times\{0\}\subset V\times\{0\}\times(a,b)\subset V\times\mathbb{R}\times(a,d)$$

$$N\cong N\times\{1\}\subset V\times\{1\}\times(c,d)\subset V\times\mathbb{R}\times(a,d)$$

and, using the coordinate in $|\Delta^1|_e\cong\mathbb{R}$,

$$P_{diag}\subset V\times\mathbb{R}\times(c,b)\subset V\times\mathbb{R}\times(a,d)$$

. However, note that the extra copy of $\mathbb{R}$ introduced above is not necessary: let

$$\bar{D}=(\{0\}\times(a,c])\cup D^{c,b}\cup(\{1\}\times[b,d))\subset\mathbb{R}\times(a,d).$$

Then the projection onto the second coordinate induces a diffeomorphism $\bar{D}\cong(a,d)$. Thus, composing the embedding of the submanifold into $V\times\mathbb{R}\times(a,d)$ with the projection onto $V\times(a,d)$ still is an embedding:

$$M\cup_{P_{diag}}N\hookrightarrow V\times(a,d).$$

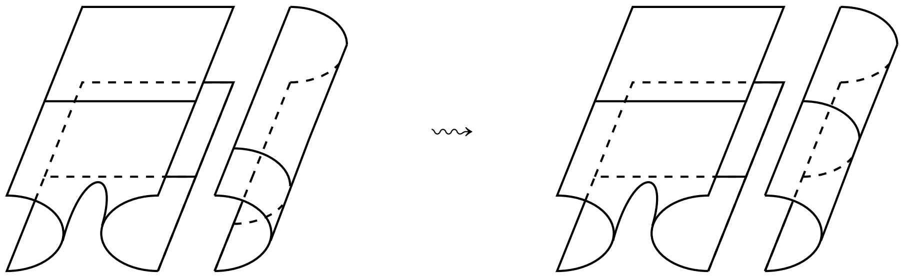
The same construction works for l-simplices: the same argument goes through with $(M,\underline{I})$ and $(N,\underline{J})$ now being l-simplices, and thus submanifolds of $V\times(a,b)\times|\Delta^l|_e$ and $V\times(c,d)\times|\Delta^l|_e$, respectively, and P a submanifold of $V\times(c,b)\times|\Delta^{l+1}|_e$. Moreover, since the shape D was chosen once and for all, this construction commutes with the spatial structure and indeed gives a map of spaces

$$glue:(\mathrm{PBord}_n^{(n-d),V})^{m,l}_{\bullet,\dots,\bullet}\longrightarrow(\mathrm{PBord}_n^{(n-d),V})_{\bullet,\dots,\bullet,k_i,\bullet,\dots,\bullet}.$$

We claim that this is a deformation retract of $\gamma^{m,l}$: Indeed, $glue\circ\gamma^{m,l}$ is the identity, since the path between the source and target in the image of $\gamma^{m,l}$ is constant. As for the other composition $\gamma^{m,l}\circ glue$, this sends a pair of elements (or l-simplices) $(M,\underline{I})$ and $(N,\underline{J})$ together with a path h from the target to the source to a pair $(\tilde{M},\underline{I})$ and $(\tilde{N},\underline{J})$ which is *not* the original one (In fact, the latter pair has a constant path $\tilde{h}$). However, there is a homotopy from $\gamma^{m,l}\circ glue$ to the identity as follows: for $\tau\in[0,1]$, send $(M,\underline{I}),(N,\underline{J}),h$ to the the following pair: modify the above construction by using $D_{s,\tau}$ and $D_{t,\tau}$ instead to obtain $P^{s,\tau}_{diag}$ and $P^{t,\tau}_{diag}$. Now one can glue M with $P^{s,\tau}_{diag}$ and N with $P^{t,\tau}_{diag}$ and embed each as above to obtain $(M_\tau,\underline{I})$ and $(N_\tau,\underline{J})$. A path h_τ between their target and source is given by the restriction of P to (i.e. the preimage of) the part between $D_{s,\tau}$ and $D_{t,\tau}$. For $\tau=0$ this is the identity map, and for $\tau=1$, this is exactly $\gamma^{m,l}\circ glue$.

"Rescaling" – the bottom and right vertical maps in* (2.2.1) *are weak equivalences: Both maps are part of a deformation retraction. Let us describe the right vertical map first.

The idea of "rescaling" is illustrated in the following picture for $n=2$, $i=1$, $l=m=1$, and $k_2=2$. Note that we just depict the cutting lines, not the intervals around them. The rescaling is performed on the right hand piece.



The deformation retract is given as follows: we observed above that the canonical map

$$\mathrm{Int}_\bullet \cong \mathrm{Int}_\bullet \underset{\mathrm{Int}_\bullet}{\times} \mathrm{Int}_\bullet \longrightarrow \mathrm{Int}_\bullet \underset{\mathrm{Int}_\bullet}{\overset{h}{\times}} \mathrm{Int}_\bullet$$

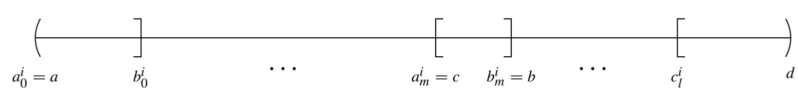
level-wise has a deformation retraction. We will lift this to the desired deformation retraction.

An element (or l-simplex) in the right hand side is given by a triple $(\underline{I},\underline{J},h)$, where h is a 1-simplex (or $(l+1)$-simplex) from $\underline{I}$ to $\underline{J}$, which we denote by $\underline{I} \to |\Delta^1|_e$. The later determines a family of diffeomorphisms $B(\underline{J}) \to B(\underline{I}(s))$ and we send a triple $\big((M,\overline{I}),(N,\overline{J}),h\big)$ to a triple $\big((M,\overline{I}),(N_s,\overline{J}_s),h_s\big)$, where $(N_s,\overline{J}_s)$ is given by the composition

$$N \subset V \times B(\overline{J}) \to V \times B(\overline{I}(s)).$$

We need the family of diffeomorphisms to have the following property: if for every $s \in [s,1]$, the cardinality $|I_j(s) \cap I_{j+1}(s)|$ is 0 or 1, then $b_j(1) \mapsto b_j(s)$ and $a_{j+1}(1) \mapsto a_{j+1}(s)$. Such maps are easily defined in a piece-wise linear way.

As for the horizontal map, the rescaling in the ith direction, let $B(\underline{I}^i) = (a,b)$ and a^i_j and b^i_j the left and right endpoints of I^i_j; and $B(\underline{J}^i) = (c,d)$ and c^i_j and d^i_j the left and right endpoints of J^i_j. Similarly to above, by rescaling $(N,\overline{J})$, we can assume that we have rescaled the embeddings and intervals such that $I^i_m = J^i_0 = (a^i_m, b) = (c, d^i_0)$, and along the path this interval stays constant. This assumption implies the the intervals can be "glued" (or rather, concatenated) to obtain an element in Int_{k_i}.



Similarly to above, this can be implemented using a deformation retraction of $\mathrm{Int}_m \underset{\mathrm{Int}_0}{\times} \mathrm{Int}_l \to \mathrm{Int}_m \underset{\mathrm{Int}_0}{\overset{h}{\times}} \mathrm{Int}_l$, which is lifted to one of the inclusion.

For every i and every $k_1,\ldots,k_{i-1}$, the $(d-i)$-fold Segal space $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$ is essentially constant. We show that the degeneracy inclusion map

$$(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_{i-1},0,0,\ldots,0} \longrightarrow (\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$$

admits a deformation retraction and thus is a weak equivalence.

Consider the assignment sending a pair consisting of $t \in [0,1]$ and an l-simplex

$$\Big(M \subset V \times B(\underline{I}(s)), \big((\underline{I}^\beta(s))_{1\leqslant\beta<i}, (a_0^i(s), b_0^i(s)), (\underline{I}^\alpha(s))_{i<\alpha\leqslant d}\big)_{s\in|\Delta^l|_e}\Big),$$

in $(\mathrm{PBord}_n^{(n-d),V})_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ to

$$\Big(M \subset V \times B(\underline{I}(s)), \big((\underline{I}^\beta(s))_{1\leqslant\beta<i}, (a_0^i(s), b_0^i(s)), (\underline{I}^\alpha(s,t))_{i<\alpha\leqslant d}\big)_{(s,t)\in|\Delta^l|_e\times[0,1]}\Big),$$

where for $\alpha > i$ and every $0 \leqslant j \leqslant k_\alpha$,

$$a_j^\alpha(s,t) = (1-\varepsilon(t))a_j^\alpha(s) + \varepsilon(t)a_0^\alpha(s),$$
$$b_j^\alpha(s,t) = (1-\varepsilon(t))b_j^\alpha(s) + \varepsilon(t)b_{k_\alpha}^\alpha(s)$$

for a smooth, increasing, bijective $\varepsilon : [0,1] \to [0,1]$ with vanishing derivative at the endpoints. This is a homotopy $H : [0,1] \times (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n} \to (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$ exhibiting the deformation retract[g]. Note that $B(\underline{I}(s,t)) = B(\underline{I}(s))$ for every $t \in [0,1]$. Moreover, for $t = 0$ we have that $I_j^\alpha(s,0) = I_j^\alpha(s)$ and the l-simplex is sent to itself. For $t = 1$ we have $I_j^\alpha(s,1) = (a_0^\alpha(s), b_{k_\alpha}^\alpha(s))$, so the image lies in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,0,\ldots,0}$.

It suffices to check that for every $t \in [0,1]$ the image indeed is an l-simplex in $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$. Since $(M, \underline{I}(s)) \in (\mathrm{PBord}_n^V)_{k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_n}$, this reduces to checking

> ***For every $i < \alpha \leqslant d$ and $0 \leqslant j \leqslant k_\alpha$, at every $x \in p_{\{\alpha\}}^{-1}(I_j^\alpha(s,t)_{s\in|\Delta^l|_e})$, the map $p_{\{\alpha,\ldots,n\}}$ is submersive.***

Since in the ith direction we only have one interval, we have that $p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s\in|\Delta^l|_e}) = M$, so in particular, $p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s\in|\Delta^l|_e}) \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s,t)_{s\in|\Delta^l|_e})$. Therefore, condition ((3)) in 2.2.7 on (M) for i implies, that $p_{\{i,\ldots,n\}}$ is a submersion in $p_{\{i\}}^{-1}((a_0^i(s), b_0^i(s))_{s\in|\Delta^l|_e}) = M \supset p_{\{\alpha\}}^{-1}(I_j^\alpha(s,t)_{s\in|\Delta^l|_e})$, so $p_{\{\alpha,\ldots,n\}}$ is submersive there as well. □

[g]To be precise, we take $t \in |\Delta^1|_e$ and extend the assignment so that it is constant outside $[0,1]$.

So far the definition of $\mathrm{PBord}_n^{(n-d),V}$ depended on the choice of the vector space V. However, in the bordism category we would like to consider all (not necessarily compact) n-dimensional manifolds. By Whitney's embedding theorem any such manifold can be embedded into some finite-dimensional vector space V, so we need to allow big enough vector spaces.

Definition 2.2.21. We define $\mathrm{PBord}_n^{(n-d),V}$ to be the homotopy colimit of d-fold Segal spaces[h]

$$\mathrm{PBord}_n^{(n-d)} = \varinjlim_{V\subset\mathbb{R}^\infty} \mathrm{PBord}_n^{(n-d),V} = \mathrm{hocolim}_{V\subset\mathbb{R}^\infty} \mathrm{PBord}_n^{(n-d),V}.$$

The last condition necessary to be a good model for the (∞,d)-category of bordisms is completeness, which $\mathrm{PBord}_n^{(n-d)}$ in general does not satisfy. To see why this is consider PBord_n, we observe that

$$(\mathrm{PBord}_n)_0 \simeq \varinjlim_{V\subset\mathbb{R}^\infty} (\mathrm{PBord}_n)_0^V \simeq \varinjlim_{V\subset\mathbb{R}^\infty} \mathbb{R}\times\Psi_0(V)$$

is a classifying space for closed manifolds of dimension $(n-1)$ (this follows from general position arguments: as the dimension of the vector space V grows, the embedding spaces $\mathrm{Emb}(M,V)$ become highly connected, so the homotopy type of the quotients $\mathrm{Emb}(M,V)/\mathrm{Diff}(M)$ become good approximations to the classifying spaces $\mathrm{BDiff}(M)$). By contrast, invertible 1-morphisms in the homotopy category hPBord_n are given by invertible bordisms between $(n-1)$-manifolds. An invertible bordism $B: M\to N$ arises from a diffeomorphism of M with N if and only if B is diffeomorphic to a product $M\times[0,1]$. If $n\geqslant 6$, the *s-cobordism theorem* asserts that this is equivalent to the vanishing of a certain algebraic obstruction, called the *Whitehead torsion* of B. Since there exist bordisms with nontrivial Whitehead torsion, the Segal space PBord_n is not complete for $n\geqslant 6$ [Lur09c].

However, we can pass to its completion Bord_n.

Definition 2.2.22. The *(∞,d)-category of n-bordisms* $\mathrm{Bord}_n^{(n-d)}$ is the d-fold completion $\widehat{\mathrm{PBord}}_n^{(n-d)}$ of $\mathrm{PBord}_n^{(n-d)}$, which is a complete d-fold Segal space.

[h]Note that the identity map from the model category of d-fold simplicial spaces to the model category of d-fold Segal spaces is a left adjoint (since it is a localization) and therefore preserves homotopy colimits. Thus, the homotopy colimit can be computed in d-fold simplicial spaces.

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